ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES

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LOGIQUE ET MÉTHODOLOGIE

Exposés publiés sous la direction de

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III

LOGIC OF ALGEBRA

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BIOLOGIE GÉNÉRALE EN RAPPORT AVEC LA CYTOLOGIE

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Professeur à l'Université de Rome
PHILOSOPHIE ET HISTOIRE DE LA PENSÉE SCIENTIFIQUE

CATALOGUE SPECIAL SUR DEMANDE

TABLE OF CONTENTS

IMPRIMERIE BARNÉOUD
— LAVAL (FRANCE) —

D. Hilbert and P. Bernays. — Grundlagen der Mathematik. Band I. Berlin, 1934.

O. Hölder. — (1) Die Arithmetik in strenger Begründung, Leipzig, 1914; (2) Die Mathematische Methode. Berlin, 1924.

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J. Tannery. — Leçons d'Arithmétique.

R. Wavre. — (1) Y a-t-il une crise des mathématiques? *Revue de Métaphys.*, 31 (1924); (2) Logique formelle et logique empiriste. ib. 33 (1926); (3) Sur le principe des tiers exclus, ib. 33 (1926).

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the casuistry of the supreme code. Thus the formalist attitude corresponds to the theological outlook.

The realist accepts the community of facts as ultimate, and for practical human purposes tries to group them by following their actual affiliations and lines of demarcation. In this scheme, mathematics occupies a rather privileged position on the ground that it coordinates our most general, and most superficial, experiences on the formation of collections and groups, and that its technique of calculation is needed in precise forecasting.

results of such an analysis into the language of a more or less general scheme of handling statements in order to transmit to mathematics the cohesion of this meta-theory.

If such a meta-mathematics, like Hilbert's Beweistheorie, grows out of the concrete analysis of mathematical processes, it may be useful in making us more conscious of the peculiarities of constructions and inference in mathematics; but in this case the validity and cohesion of the super-theory will be in fact derived from the underlying mathematical facts. This validity and cohesion must be derived from *some* source, from some *de facto* experience, as even the basic assumptions of any logic of statements are founded on our experiences in handling meaningful statements in the « right » way where « right » means conform with their meaning and with that of the proposed handling. Even Russell's general conception of implication (ironically called material) is based on the *practical* requirement that true premises should not lead to false conclusion.

The position for formalist proofs of consistency is therefore this : In any theory of inference, however abstract, a distinction must be drawn between a right and a wrong way of inference. An arbitrary distinction would lead to a meaningless concrete game with symbols, isolated from science proper, which would not be considered as a *super*-theory. If the distinction is based on our actual way of handling statements, the validity of the super-theory results from the examples which the theory wants to probe. Abstractions have no power by themselves; even their own interrelations are determined by their meaning. A thoroughgoing formalist foundation of mathematics is impossible, and the efforts in this direction were mostly helpful in leading to a better understanding of the relation between logic and mathematics; in particular, to a deeper insight into the function of the principles of logic.

The ultimate source of formalist attitude is the idea of subsumption, the desire to create a statical hierarchy headed by the Absolute, here absolute logic, conceived as the sovereign power, independent of the lower grades, from which law and order emanates. Only laws traced to this supreme source are really admitted and actual relations on the lower grades have to be « justified » by

numbers themselves) is of fundamental importance for our tracing the flow of this transmission into the structure of mathematical analysis, i. e. Calculus. We shall discuss the corresponding problems in another tract of this collection.

33. — *The formalist attitude.* — Formal consistency in a restricted system S′ of statements, like arithmetic or algebra, means that (i) the initial statements (axioms) do not contradict or exclude each other, (ii) the inference used in the system to derive statements from other statements do not lead to contradictory or contrary statements. To establish consistency in S′ we have to « deduce » those two properties from some admitted statements by some admitted method of derivation, i. e. we have to start with an initial system S endowed with a method of inference. Therefore a formal proof of consistency is necessarily relative; it amounts to showing that if S is consistent, so is S′. Thus unless the consistency of S is obvious (?) or at least more certain than that of S′, the reduction only has a theoretical interest.

The initial statements on integers are considered by formalists of the Hilbert type as undeniably true and thus consistent in themselves so that we only have to analyse and probe mathematical inference. The system of comparison S is some kind of logic.

The logic of collections, including classical logic, applies to some mathematical statements like inequalities between integers but exact numerical relations cannot be verified by that logic, or by any logic S unless it initially contains all the details of mathematics. In the latter case however the special problem of consistency of mathematics dissolves into that of S, and this problem requires a new super-system.

Also mathematical induction refuses to be reduced to any finite, statical system; and if we include it into S, it escapes probing. If mathematical induction is included in the super-system S, the formalist attitude comes very near to realist attitude as to the problem itself, which reduces in fact to find the link between more and more distant parts of mathematics. As to the method, the realist thinks that a sound analysis of the constructions and processes in mathematics described in ordinary language (plus ordinary mathematical symbols and a few logical symbols, may be) will suffice, whereas the formalist insists on transcribing the

fore the definiteness and consistency of integers are automatically transmitted to real algebra only in an evolving sense viz. to arithmetical problems on definitely given real numbers.

Even the attribute definition of real numbers acquires some kind of definiteness from its connection with integers. In this definition, real numbers are systematically and simultaneously specified by the enumeration, at every step, of all the ten possible choices for the digits. At every step this process only produces a finite number (ten times the previous number) of rational numbers all considered as unfinished, so that it never completes the construction of a single real number; still, the process itself possesses a determinate character as opposed for example to the vague « like this » definition of functions by an appeal to some examples on one-one correspondence.

In fact, this attribute definition of real numbers is (i) constructive and the steps form a devolving sequence of simple arithmetical operations (adding a digit on the right), (ii) the links of this chain are held together by the thread of integers; i. e. no notions foreign to integers are used, and the notions based on integers are used in the proper way (as in arithmetic). Still, however definite the species produced by this process may be, the elements themselves, i. e. the individual real numbers, are not definitely determined by it, since we are not told which digit to take at the various steps to get the individual number in view. Thus the real numbers are only definite collectively, not individually, and an individual real number needs a special devolving process that contains no indetermination.

The direct experience or intuition of continuity is not very helpful for the conception of an individual real number. In fact, when we try to go through the numbers of an interval in their order of magnitude, continuity dissolves into the fluidity of a change where one individual *becomes* another, and it is this change, this becoming, that we experience, and not the separate individuals. The collective determination of real numbers seems to correspond to this intuition of continuity which is rather natural in a characteristica constructed for the description of our experiences on continuity.

The fact that the definite and consistent character of integers is transmitted to the definition of real numbers (though not to the

between *a* and *b*. Thus in logic we only have $a < b$, say, whereas in mathematics we determine the exact difference *d* so that we should be able to put $a + d = b$. *When we know d*, we can, of course, interpret the result in terms of equality between collections; but in mathematics the essential processes are the determination of sums, difference, quotients, etc.

This divergence between logic and mathematics becomes more obvious when we pass to fractions where we have to solve a practical problem viz. that of finding an adequate symbolism for the notion of ratio so that our symbolic operations should correspond to real properties of ratios. We construct this symbolism in terms of integers because this method suits the case as well as ourselves, and the unified system of rational numbers that includes integers, in a rather awkward, multiform shape, helps by its economy.

The exclusive use of integers in such a construction is specially helpful to make our new problems definite and our new symbolism consistent. For instance $\frac{a}{b} + \frac{c}{d}$ is a definite problem, for $ad + bc$ and bd are such; the order of magnitude of $\frac{a}{b}$ and $\frac{c}{d}$ is definite, for the question of whether $ad - bc$ is positive, negative or o is definite. Also, since the new notions and methods are shorthand symbols for statements and methods in the arithmetic of integers, any contradiction in the arithmetic of fractions would be *ipso facto* a contradiction in the arithmetic of integers. The methods of proving statements on fractions are created alongside with the algebra of fractions; they form part and parcel of this algebra, and these methods are justified by the fact that every step in the construction of these new notions and new methods is an admitted definite construction on integers. This shows that the algebra of fractions is as consistent as the notion of integers. For the actual construction see Stolz-Gmeiner or Hölder (**1**).

The same reduction process applies to the algebra of positive and negative rational numbers but stops short of real numbers. It is true that *when* a *and* b *are given by definite rules*, $a + b$, ab, $\frac{a}{b}$ etc. are definite problems, but there is no definite devolving process giving every real number one after the other so that we cannot take every real number as definite as integers are. There-

notion of whole numbers when extended to quaternions etc. undergoes a radical change leading to a better understanding of the network of relations contained in that notion. The arithmetics (note the plural!) emerging from new algebras considerably deepen our understanding of the essential features of arithmetic.

This dynamical nature is as also manifest in the fundamental ideas and constructions of logic itself as shown by new developments of the last fifty years. In particular, the unterstanding of the *relative* character of the three principles by the examination of their function in scientific theories and especially in mathematics is bound to lead to a better understanding of their nature.

The illusion of a final statical state comes from the fact that in our actual work within a given system we have to take this system as it is presented to us and thus we are inclined to isolate it from its history and from its roots in reality. In such an isolated system the only dynamical aspect is the drawing of conclusions and, for practical purposes, even this is mechanized as far as possible.

32. — *The problem of consistency in algebra. The realist attitude.* — Our fundamental experience is that our world consists of more or less definite things more or less definitely interrelated among themselves. In particular, these things appear in groups and the groups of definite objects like rigid bodies possess certain general properties. Both logic of collections and mathematics try to describe some of these properties from two slightly different angles, and there is no reason why one of them should not make use of the results of the other. For ex. in L. C. we used the fact that $2^7 = 128$, and in the extension of some results to « n » collection variables we have to use mathematical induction. In mathematical arguments, on the other hand, we make use of rules as given in the theory of inference of L. C. This collaboration needs no justification, for the two theories come from the same source.

In mathematics syllogisms are seldom used explicitly though a good many arguments specially on integers could be couched in syllogisms. For instance when we conclude from $a < b$ and $b < c$ that $a < c$, the fundamental collection interpretation leads to Barbara. The main reason for not using syllogisms is that L. C. is not concerned with the precise numerical relations

tions, like the axioms of geometry, obtained from facts by an extrapolation process, i. e. by hypothetically extending the sphere of validity of certain observed relations. This extrapolation is indispensable for the organization of our rather chaotic experiences into more or less coherent wholes called theories. In fact, if, under the pressure of our needs, we want to guess how events will happen in the future, we must guess their habits, and if in this work we want to use our intellect, we must formulate these habits in statements. The statement of a habit goes beyond actually observed facts but it is based on facts.

Since thories are made by the help of our intellect for its own use, they display features that reflect the characteristics of intellect and therefore are not necessarily shared by other processes in the outside world. To direct our actions by forecasting facts a theory must be decisive, it has to say yes or no even if only hypothetically as a guess. Hence the requirement of consistency. This requirement is however not absolute. As a matter of fact in Physics we make use of theories like classical dynamics and atom mechanics that contain mutually exclusive statements. We may deplore this state of affairs but we accept it on the ground that these theories are based on overlapping fields of observed facts, and they only differ in their way of extrapolating, completing their respective fields of facts into abstract models.

Thus in scientific theories the *principle of contradiction becomes a problem*, viz. *that of consistency*, and there are as many special problems of consistency as there are attempts to group facts into coherent models. The work involved in the construction of a few such systems of statements is so huge and the special problems of consistency are so hard, that for a scientist any attempt to organize all statements once for all into coherent blocks appears as utterly futile, and, in view of constant alterations, also unnecessary. In scientific practice the static idea of Russell's hierarchy of statements, corresponding to the Platonic pyramid of ideas, is humbly replaced by attempts to construct a few coherent systems subject to radical alterations and development. Also fundamental ideas like space, time, causality are not exempt from proper development as relativity theory and atom mechanics have proved.

Ideas are not static even in mathematics. For ex. the original

is not definite. I cannot actually add k_1 to 1937; therefore sums of small integers are indefinite. If you want a number you are not likely to know, replace myself in the example by humanity, or take doomsday.

In my opinion we have to separate known facts from unknown ones for the practical progress of knowledge. Certain mathematical problems can only be solved by solving a good many problems beforehand, i. e. the strategy of scientific research requires this distinction, but it also requires the knowledge that the definite character of certain problems makes other problems definite. From the point of view of action Brouwer's distinction points to the past (present included) whereas the transmission of definiteness points to the future.

The general attitude as presented in this tract towards the use of the principle of excluded middle in mathematics is that of Brouwer's, we do not admit the indiscriminate use of this principle and we query Hilbert's assumption that the answer to every mathematical question is yes or no even if it is « well put ». Our attitude however is not based on Brouwer's distinction between known and unknown results, but on the fact that every restricted field of inquiry is indifferent to most questions and thus we cannot *a priori* exclude this possibility even in the case of carefully prepared questions apparently within the compass of the field. The only way out seems to be to find a thread of Ariadne in the maze of problems leading from a definite set of problems on integers to other problems made definite by their connection with the first set.

31. — *Principle of contradiction. Consistency.* — Our ultimate test for truth is experience, practice. Certain statements like « this chair is brown », « there are three persons in this room ». « There are various things in the world » etc., can be finally tested; others including measurement like « the sum of the angles of a triangle is 180° » can only be tested approximately, so that if we substitute $\left(180 - \frac{1}{10^{1000000}}\right)^0$ for 180° our facts will support both statements equally well, and later experiences may show that the sum of angles depends on the size of the triangle, so that both may turn out to be false.

Statements of the latter kind appear as consequences of assump-

Brouwer who first emphasized the difference between mathematical facts and the language we create to describe some such facts. This amounts to admitting an essential difference between the objective character of certain mathematical facts and the accidental character of our knowledge of them. I also agree with Brouwer that not *all* mathematical facts are objective and that is why I want a test for objectivity. For such a test (i) we have to point to some undeniably definite, objective mathematical facts like the elementary properties of integers, (ii) we have to show that this objective character can be and is transmitted to certain mathematical problems. In this way certain problems will be made as objective (definite) as the elementary problems on integers.

Since for π the transmission works, it is a definite question whether in π there are groups 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 of consecutive digits. Hence, if we denote by k_n the place number of the first term in the kth group, the sequence $k_1, k_2, \ldots$, is a definite non-decreasing sequence of integers about which we know precious little, since we do not even know if it is empty or not.

Putting

$$c_m = \left(-\frac{1}{2}\right)^{k_1} \text{ if } m > k_1, \text{ and } c_m = \left(-\frac{1}{2}\right)^{m} \text{ otherwise,}$$

we obtain a sequence whose limit is (i) 0 if there is no k_1, (ii) $\left(\frac{1}{2}\right)^{k_1}$ if k_1 exists and happens to be even, (iii) $-\left(\frac{1}{2}\right)^{k_1}$ if k_1 is odd. Since at present we can exclude none of these three cases, we do not know whether this limit is positive, zero or negative.

Brouwer draws the conclusion that the continuum is not an ordered set since there is a number, viz. the limit of c_m, whose place in that order is not definite. From the fact that we actually do not know, and most likely will never know, whether the sequence of k_n is empty, finite or endless, Brouwer concludes that the question « is every set of real numbers either finite or infinite » is indefinite. I agree with this conclusion but I cannot accept the argument.

Here is a parallel example. Denote by k_1 the number of years I still have to live ; k_1 is a small integer and I do not know its place in the series of integers. Therefore the order of integers

here a chain for the transmission of definiteness from problem to problem, and the chain continues unbroken down to the series of integers, so that the ultimate source for definiteness is always the same viz. the definite character of integers.

30. — *The principle of excluded middle.* — The logic of collections is based on the static idea of definite groups of definite objects. In L. C. we do consider common or distinct parts of collections, but the dynamical aspect of integers displayed in the unfinished, endless character of the passage from integers to larger integers is alien to L. C. proper. This aspect introduces a new type of inference, mathematical induction, for finding common properties of devolving sequences like the properties

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \quad \text{and} \left(1+\frac{1}{n}\right)^n < 3, \textit{ for every } n.$$

From these inductive properties we see that the rational numbers $\left(1+\frac{1}{n}\right)^n$ form an increasing and bounded sequence, so that by the process described in section 28, they determine a decimal form called e. The essential point in mathematical induction is to show that in a given sequence a certain property, like < 3, or a certain relation like $a_n < a_{n+1}$ is transmitted from the numbers already constructed to the collection enlarged by some new members.

In general, if for a given sequence x_n of real numbers we have

$$x_n < x_{n+1} \text{ and } x_n < B$$

the method proves that there is a definite decimal form (real number) x associated with the sequence. This method works in particular for infinite series of given positive real numbers and thus we consider $\sqrt{2}$, e, π etc. as definite real numbers in spite of the fact that actually we only know some hundreds of their decimal figures, and we entertain no hope of having them all determined in any distant future. It follows that we consider the answer to the question « is the millionth decimal figure of π equal to 3 ? » as determinate (yes or no) although actually we are unable to decide the issue.

The same logical difficulty arises in the case of sums of large integers when actually we cannot compute the sum. It was

that the idea of complex number, say, is an attribute idea, and its definiteness as an attribute is derived from the A-definiteness of the idea of real number.

The notion of operation requires some comment. In an expression like $2x^2 - 3x + 5$ the operation itself viz. $2(\ )^2 - 3(\ ) + 5$ is a clear and concrete idea. The fact that it can only be performed when a definite x is given does not detract from its concrete nature. Also the joint effect (sum) of two such operations can be determined without any given x. For ex.

$$2(\ )^2 - 3(\ ) + 5 + 4(\ )^2 + 2(\ ) - 7 = 6(\ )^2 - (\ ) - 2.$$

Therefore we consider polynomials and their quotients as definite operations as soon as their structure as operations is definite, although « polynomial » is an attribute idea defined by examples and by suitable restrictions.

The theory of simple equations up to quartics needs qualification by adding that the formulæ and methods for the determination of their roots only work when the coefficients are definite, or else that we only know that those rules cannot fail us, which is the all important point. The fundamental theorem needs a careful analysis (see Brouwer (5)).

For the logical analysis the table of derivation for the seven elementary operations of algebra is rather useful.

$$\begin{array}{ccccccc} \text{counting} & \rightarrow & \text{sum} & \rightarrow & \text{product} & \rightarrow & \text{power} \\ & & \downarrow & & \downarrow & & \swarrow\ \searrow \\ & & \text{difference} & & \text{quotient} & & \text{root, logarithm} \end{array}$$

The direct operations like product and power are particular cases of the parent operation, and their inverses are obtained when one of the two constituent numbers and the result are given and we have to find the other constituent number. These inverse operations are carried out by a series of more or less systematic trials. For ex. at every step in a long division we have to find q and r such that $a = qb + r$, and both a and b are small integers. In the extraction of roots the steps form a devolving sequence of groups of simple trials every one of which is a multiplication. In the case of logarithm where we have to solve $a^x = b$, the trials consist in raising a to fractional powers made definite by the previous problem. Thus we have

and

$$A_n^2 < 2, \qquad B_n^2 > 2$$

are given by the previous work. For every n the arithmetical work involved is the same, viz. at most five trials like (1), i. e. five multiplications. Therefore if the previous work is considered as definite, A_n and B_n will be definite; and then, unless we restrict multiplication to small numbers, we have to accept also the nth step as definite. But the first two or three steps are certainly definite so that they form a suitable peg to hang a chain of definite operations on.

Problem 7 involves such a devolving sequence of definite operations and thus we can accept the whole process that determines A, consequently also A, as definite, although we may not be able to calculate A to any degree of approximation. As a matter of fact A is related to S in a rather mechanical way. As the sequence S devolves according to the given rule, the integral parts, after a finite number of jumps upwards, remain the same, and *then* the first decimals begin their regular jumps upwards until after at most nine jumps also the first decimals remain the same, and so on. And A is constituted by the stationary part that embraces more and more decimal figures.

29. — *Operations in algebra.* — The results of the preceding section readily extend to products and to the two inverse operations. We repeat however that all decimal forms have not been admitted as separate, definite objects; only relatively few can be made definite by concrete rules. Therefore $a + b = b + a$ only means that the sum of two *definite* real numbers is commutative, which is the case, for, at every step, the two processes that define $a + b$ and $b + a$ are identical. We might also say that $a + b \neq b + a$ does not or cannot actually occur; and this in reality is quite sufficient for practice and for theory. This remark extends to the other rules of manipulation.

No new logical problem is involved in the extension of the idea of number to complex and hypercomplex numbers, or even numbers to an infinite base like tensors or infinite matrices with their appropriate rules of manipulation. It is only when the base goes beyond the reach of integers as in matrices to a continuum base that we need to reconsider the position. We notice however

(ii) It follows that there is a last jump in S and any of the numbers (all equal) after the last jump is the greatest.

Part (i) is based on general, admitted properties of integers and on the definite character of the rule that produces the sequence. Part (ii) is also based on this definite character which involves that the jumps are at definite places though we may not know where. The rule of construction for S, if definite, makes these places as objective as the rule of construction for integers makes *their* places definite, irrespective of our knowing them. Therefore the last jump is definite.

Therefore if in problems 1, 3, 5 « find » means the concrete knowledge of the number or at least the knowledge of a rule for constructing or finding this number, these problems can only be tackled in exceptionally simple cases. Thus a thorough-going constructionist would reject the proposition that « in every non-decreasing and bounded devolving sequence of integers there is a greatest » but on the same ground he ought to reject the same general property of a finite but very long (not monotone) sequence of integers. In both cases it is a machine, the rule, that does the work for us; and if an object like this machine is definite, its properties and its work are definite too. Since the machine can do more than we can, and this was the very reason why we created it, it is not so surprising that a machine produces definiteness where we cannot.

In a similar way in problems 2, 4, 6 we change « find » into « there is » and we accept them as definite if the rule producing the integers is definite.

As to problem 7 we remark that apart from the numerical data the successive problems that constitute the rule of construction for A are all the same, and the numerical data are furnished by the preceding problem and by the given sequence. To make this point clearer take the rule of construction for $\sqrt{2}$. At the nth step we have first to decide whether

$$(1) \qquad \left(A_n + \frac{5}{10^{n+1}}\right)^2 > 2 \text{ or } < 2 \ (= 2 \text{ is not the case})$$

where

$$A_n = 1.a_1a_2 \ldots a_n, \qquad B_n = A_n + \frac{1}{10^n},$$

1. Find the greatest of the integers a_{10}, a_{20}, ..., a_{n_0}, ... and denote it by A_0.

2. Find a number of (S) whose integral part is A_0 and if a_{n_0} is that number, denote by S_0 the sequence a_{n_0}, a_{n_0+1}, a_{n_0+2}, ... We notice that the integral part of all these numbers is A_0.

3. Find the greatest first decimal figure in (S_0) and denote it by A_1. We notice that the integral parts being all equal to A_0 the first decimal figures cannot decrease (since S_0 is a non-decreasing sequence) and they are all $\leqslant 9$.

4. Find a number in S_0 whose first decimal figure is A_1 and if a_{n_1} is that number, denote by S_1 the sequence a_{n_1}, a_{n_1+1}, a_{n_1+2}, ... We notice that all these numbers begin like $A_0 \cdot A_1$, ...

5. Find the greatest second decimal figure A_2 in S_1.

6. Find a number a_{n_2} of S_1 whose second decimal figure A_2 and denote by S_2 the sequence

$$a_{n_2},\ a_{n_2+1},\ a_{n_2+2},\ \ldots.$$

We notice that all these numbers begin like $A_0 \cdot A_1 A_2$ ....

7. Proceed indefinitely in the suggested way.

The number $A = A_0 . A_1 A_2 A_3$ ... so constructed is said to be the number associated with the sequence (S), and $A + B$ and AB are then defined as the numbers associated with the sequences $A_n + B_n$ and $A_n B_n$ respectively.

For the analysis of the difficulties inherent in such a construction, we divide the constituent problems into three groups (1, 3, 5), (2, 4, 6) and (7), and for the first group we consider a devolving non-decreasing sequence S of integers, all less than 100, say. Taking them in turn we may find that the billionth is 59, say, and unless we hit on 99 this direct method of inspecting the numbers of S one by one will never tell us which is the greatest of the lot. How do we know then that there is a greatest among them?

To point out the greatest involves more knowledge than the rather vague information that there is a greatest. For instance we know that one of two consecutive integers is certainly odd without knowing which one, and we consider such information as valuable. Thus we argue like this.

(i) 100 or more jumps in a sequence like S lead to a number 100. Therefore there are less than 100 jumps in S.

definite objects. A-definiteness is mostly derived from the definite character of the general properties of collections as formulated in the logic of collections and from the definite character of arithmetic. The chain of transmission of definiteness from construction to construction is one of the characteristic features of mathematics. We shall illustrate these general remarks by a careful analysis of a single problem in the construction of real algebra, viz. that of the sum of two real numbers (See Dienes (4) p. 2-3).

28. — *Sums of two real numbers.* — The sum of two integers is obtained by a chain of simple additions, where every operation is determined by the result of the preceding one and by two digits. Since, in the construction of the characteristica for continuity, measurement suggests the extension of arithmetic to decimal form, we naturally try to extend the technique of addition to our new numbers. For this purpose take two concretely given decimal forms

$$a = a_0 \cdot a_1 a_2 \ \ldots, \quad b = b_0 \cdot b_1 b_2 \ \ldots$$

and put

$$A_n = a_0 \cdot a_1 a_2 \ldots a_n, \quad B_n = b_0 \cdot b_1 b_2 \ldots b_n$$

From the technical point of view the sum $A_n + B_n$ involves the same mathematical operations as the sum of the two integers we obtain from A_n and B_n when we disconsider their decimal points. At every step

$$A_n + B_n < a_0 + b_0 + 2$$

and thus taking larger and larger n we obtain an increasing (non-decreasing) and bounded sequence of terminated decimal forms.

In the description of the way in which this sequence determines the succesive digits of the decimal form that we call $A + B$, we take the slightly more general case of a sequence of non-decreasing and bounded decimal forms terminated or not;

$$\text{(S)} \qquad \begin{aligned} a_1 &= a_{10} \cdot a_{11} a_{12} \cdots \\ a_2 &= a_{20} \cdot a_{21} a_{22} \cdots \\ &\vdots \\ a_n &= a_{n_0} \cdot a_{n_1} a_{n_2} \cdots \\ &\vdots \end{aligned}$$

where, by hypothosesis,

$$(1) \qquad a_n \leqslant a_{n+1} < B.$$

be any of the ten digits 0, 1, 2, 3... 9. Thus, roughly speaking, « all lengths » means all decimal forms, recurring or not. Thus we are led to consider decimal forms, terminated or not, recurring or not, like .24681012...98100102...

Can we consider such a decimal form as an individual object? If there is a definite rule given for the continued determination of digits i. e. when the successive digits form a devolving sequence, we can say that such a decimal form is as definite as the series of integers, and this is good enough for a mathematician, or for anybody else.

If, however, we try to determine the successive digits by arbitrary acts of choice, we only get a finite decimal form thought of as unfinished, for we can only choose freely a finite number of times, nay, a very restricted number of times. Even for a finite but long sequence of digits we must give a rule i. e. we must find a machine to do the work for us. If we stick to free choice, the decimal form *ipso facto* is considered as indeterminate, for after every step we reserve for ourselves the right further to specify it in any (or some) of the ten different ways. Such a decimal form in *statu nascendi* may be called an *evolving* (werdende) decimal form. At every step, an evolving decimal is a finite decimal form thought of as unfinished.

An evolving decimal form is, of course, not completely indeterminate. If the integral part is *o*, we can « prove » that the product of two such evolving decimal forms is also such a decimal form, etc. As far as the evolving decimal form is determined we can make use of it. By its very nature, however, an evolving process cannot lead to an individual decimal form, it only produces more and more specified species. In logical parlance we could say that the accepted principium individuationis (object-building convention) is the existence of a definite digit at every place, and a devolving sequence of integers satisfies this requirement as completely as possible; whereas the very conception of an evolving decimal implies indetermination as a play-ground for free choice.

Definiteness can be obtained from two sources (i) from the definiteness of certain objects, (ii) from the definite character of certain attributes. In particular, O-definiteness in mathematics is derived from that of the series of integers accepted as

objects a given rule *can* create. In this respect there is no difference between devolving and envolving processes; both only create virtual existence.

They deeply differ, however, in their way of producing virtual existence. A devolving sequence produces the objects one after the other, so that at every step the rule deposits a finished article; whereas in an evolving process all the elements are constructed simultaneously bit by bit so that every step contributes to the completion of the definition of all the elements. Therefore unless we restrict ourselves to follow a single thread in the ever increasing pattern we cannot arrive at a definite element.

This difference between sequence and set is best displayed when we try to express both in terms of the idea of collection. At every step the sequence produces a definite collection of objects, so that a sequence can be described as an ever increasing collection of definite objects where the joining of objects is given by a definite rule. Thus the idea of sequence goes beyond that of collection but can be defined in terms of collections and of the idea of endless succession borrowed from the sequence of integers.

On the other hand, at every step, an evolving process produces a definite collection of species we do not want to consider as our ultimate objects, so that the process never produces a collection of elements. A definite element is only obtained if we specify the evolving process into a devolving process. Therefore the application of the idea of collection, in particular the application of their logic, to *all* the elements such an evolving process *can* produce is at least doubtful.

27. — *Real numbers.* — To anticipate events we have to know how our environment changes. The principal kind of change is motion (perhaps because we ourselves can produce it) and we conceive motion as a *continuous* type of change. Also, when a rod is heated, we assume that it cannot pass from a length l to another length l′ without passing through all possible lengths between l and l′. This is the practical, intuitive idea of continuity, and we are asked to construct its characteristica.

Length is obtained by measurement, and if a rod is slightly larger than 2.375, say, the next decimal figure may turn out to

butes bracketed together are as a rule mutually exclusive. In abstract symbols we get the scheme :

$$(a_{11} + a_{12} + \ldots) \times (a_{21} + a_{22} + \ldots) \times (a_{31} + a_{32} + \ldots) \times \ldots,$$

i. e. we have a sequence of conjunctions of disjunctive groups of attributes, and an object is « obtained » by choosing one of the attributes in every bracket. Since natural objects possess an indefinite number of attributes, only an endless enumeration may individualise an object. In the case of decimal forms every digit is chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, i. e. every step admits mutually exclusive attributes one of which is always the case.

The essential difference between natural and abstract objects defined by this process is that real objects are conceived as existing independently of our analysis of their attributes, whereas abstract objects are constructed by the enumeration of admitted attributes.

To distinguish such a systematic class from the ordinary vague idea of objects possessing a certain attribute, especially when the objects in view are objects of thought, we shall call it a *set*. Thus a set is an attribute with a given systematic specification and with a definition of its « objects », and to avoid confusion the latter will be called its elements. The process will be qualified an *evolving construction* as opposed to devolving constructions.

The difference between set and collection lies in the fact that until there is an « either-or » in the construction, the object in view cannot be considered as determinate. A few 'either-or'-s can, of course, be dealt with, but a systematic indetermination cannot be conceived as the final determination of an object. Therefore only a devolving sequence of choices will lead to a definite element that we can consider as our object i. e. a real number in the full meaning of the word. Until there is indetermination we only get species. In logical terms, every object of a collection has its own principium individuationis, whereas the elements of a set are only isolated in a collective way.

Compare now sets with devolving sequences of numbers. Even in the case of a devolving process only a few objects can be actually produced by us and the rest is left to the rule of construction, i. e. a devolving sequence of numbers consists of

of electrons, etc., the parts remain objects and are not more or less general than the system. Collections however possess a similar relationship in $a < b$. Also colour and shape cannot be arranged in order of generality i. e. there are several scales of generality. This leads to a formalism very similar to that of collections, with an inference based on the transitive property of subsumption. The two logics differ in the question of existence, which is the basic point of view in L. C. and which does not appear in the formal logic of attributes where conjunctive, combinations like a plane geometrical figure that is square *and* circular, or unicorn, are admitted irrespective of the fact that there may be no objects possessing such compound attributes.

To show the relation between the two logics we remark that the fundamental process of statement is to relate an attribute to an object, and thus only qualities of an object or of a collection of objects can form a proper logic. For instance, the logic of collection describes some general properties of collections and relations between collections; it is a kind of characteristica of collections. Mathematics is the characteristica of numbers where, due to the collection character of integers, the logic of collection applies to that extent but only to that extent. Also systematic description of the possible information about an object forms a particular logic or characteristica of this real or abstract object.

In this sense the logic of attributes is the characteristica of the idea of attribute related to no object at all. In L. C., in mathematics or in everyday life, thinking means handling arrows (symbols) that point to reality, and this actual connection with reality pours life into their abstractions, whereas in the logic of attributes quâ attributes their roots are cut off to give a phantom independence and objectivity to attributes.

26. — *Collection, sequence, set.* — The idea of class has been derived from the notion of natural objects. Take a sack of balls of different colours, shapes and materials and try to individualise the balls by a systematic classification of their attributes. We obtain the following scheme where $+$ and $\times$ stand for « or » and « and » respectively. :

(red $+$ green $+$ blue...) $\times$ (spherical $+$ egg-shaped $+$ ...) $\times$ (wooden $+$ metal...) $\times$... ; and for the objects in view the attri-

red as a species and not as an object until also the second decimal form is completely given.

Thus the general idea of real number as an endless decimal form consists in (i) admitting certain ways of specification for the attribute « decimal form » (ii) stopping the systematic specification at a definite point to obtain our specific objects. These rules however do not create *all* the infinite decimal forms (real numbers); in fact, they do not create a single one, they are only used in testing actual construction like ·222... This general idea of *real numbers as endless forms is an attribute notion.*

What is the « class » corresponding to this attribute? Apart from the few concrete specimens the individuals of this class do not exist in the full meaning of the word, or even in the devolving sense, because we have not formulated a rule for their successive construction. Since abstractions and concepts, unlike natural objects, are of our own make, they only exist when made. In this respect they are similar to human made machines like dynamos, radios etc. Therefore, the *complete class of decimal forms as a collection of seperate objects does not exist.* We notice that we accepted the existence of successive integers as a devolving sequence on the ground that a machine (add one) produces them one by one, and that we were unable to think of this production of integers as definitely stopped.

Therefore we must carefully distinguish collections from the indefinite notion of class. On the other hand, thinking in attribute, if kept apart from the notion of class, can be perfectly clear, and it is, in fact, unavoidable; so is thinking in extension i. e. in collection of definite individuals, for the very foundation of mathematics, viz. integers, are symbols for continued collections of objects. Also, combination of attributes involves a process in extension. Thus in everyday life as well as in scientific thought, thinking in intension and extension are inextricably interwoven, and mathematics furnishes a typical example for the mixture of these opposite types corresponding to the dialectical polarity between attribute and object.

An independent logic of attributes will only be needed if attributes exhibit qualities and relationships not shared by objects. Subsumption like red and coloured seems to be such a specific relation, since in systems of objects, like atoms made up

future, other planets, animals of extremely small size, and also heraldry and the world of collective imagination, i. e. we qualify the statement by restricting it to larger animals living at present on the Earth. Without such restriction we cannot say anything about the class, not even that it is empty.

This assumption of an absolute class pertaining to every attribute becomes really misleading when we extend the object-attribute polarity to our thinking process, when we consider attributes, or anything else we may think of, as objects of thought possessing attributes. Abstract attributes like prime numbers, or continuous functions, can be as clear as any attribute of concrete objects but, because of our freedom to consider any attribute as an object or as an attribute, the world of abstractions does not consist of definite objects; we have to create them by decree. In fact, piling attributes upon attributes we only create more and more complicated attributes, unless at a certain stage we declare the pile to form an object of thought.

We shall illustrate this important point by the attribute « decimal form » defined by some examples and by the admitted way of altering these samples viz. by changing their digits. Further qualifications by the size or colour of digits etc. are tacitly excluded. If we stipulate that the integral part should be *o*, we obtain a species of the genus « decimal form » by what the logicians call a specific difference. By prescribing some of the first digits, the first two say, we further qualify the species « decimal forms like ·37... », i. e. beginning with a *o* integral part and with the digits 3 and 7 ». In this way we may obtain a species of a species but not an individual object.

In arithmetic when we transform 1/2 into ·5, we consider ·5 as a definite object (of thought). In the algebra of real numbers we consider « complete » decimal forms as our specific objects like ·333... or ·1234567891011... and thus ·5 will mean ·50000..., i. e. unless all the consecutive digits have been given by a definite rule, we do not consider the decimal form as a definite object. To make this point clear take double decimals (a, b) i. e. two decimal forms in a certain order where *b* is considered as a sort of continuation of *a* so that, for instance, $(a, b) < (c, d)$ if $a < c$ (irrespective of b and d) and if $b < d$ when $a = c$. In this case completely given single decimal forms like $a =$ ·222... will be still conside

CHAPTER VI

REAL NUMBERS

25. — *Attribute, class.* — As a preliminary to the discussion of logical problems in connection with real numbers, we shall make some remarks on the intensive side of logic as a complement to the extensive side discussed in the previous section. An attribute is first perceived as a quality (property) of an object, and then the perception of the « same attribute » in different objects gives them a phantom independence. This knowledge of an attribute as defined by some samples, i. e. by the « like this » process, is in a sense absolute, for our understanding of the proposed attribute is made final by the reduction to perceptions. This corresponds to the fact that the ultimate definition of an object is to point to it (the « this » process).

Moreover this understanding of an attribute does not require, and in general does not include, the knowledge of all the objects possessing the attribute called the *class* corresponding to the attribute. Samples suffice and the notion of « all the objects possessing the attribute » is entirely alien to the nature of attributes. We remark, however, that the cognitive value of an attribute lies in the samples forming its roots that connect it with reality.

Logicians generally attribute an absolute existence to the class on the ground, I believe, that real objects exist with all their qualities whether we know them or not. Now, take the stock example of unicorns for an empty class. When we say that the class of unicorn is empty, we really mean that to our knowledge at present no such animal lives on our planet. We exclude past,

assumptions are naturally taken for + statements, no inference based on truth values can detect a clash between assumptions or between their consequences that are also taken for + statements. This means that *before* material implication can be applied in S we have to know *from other sources* that S is consistent and thus this method cannot be used for the very purpose it has been invented for.

Also, when we confront theories with an observed fact, we want to know which of the proposed theories imply this fact and which do not, although in their own sphere all these theories may be right. In fact, in every particular science, together with notions and methods, inference also grows out of the fundamental experiences (practice) that this particular science is out to enlarge and deepen. Since collections of definite objects occur in most fields, L. C. and mathematics may be useful in other sciences whose proper logic however may display new features.

Russell's logic of relations, and in particular his theory of inference called material implication, is an attempt to create an absolute logic, large enough to include mathematics. His fundamental conception can be described in the following way.

A group of two or more statements (true or false) is not a statement until a convention is given that assigns truth (T) or falsehood (F) to the group. We may agree for instance that (A, B) is true if and only if both A and B are true, and false in the other cases, (AB or A_xB). In a more generous mood we may agree that (A, B) is true if A or B is true, and false if both A and B are false, ($A + B$). If $\overset{+}{A}$ and $\bar{A}$ denote true A and false A respectively, we can assign T and F to (A, B) in 16 different ways. If we assign T to ($\overset{+}{A}$, $\overset{+}{B}$), ($\overset{+}{A}$, $\bar{B}$), ($\bar{A}$, $\bar{B}$) and F to ($\bar{A}$, $\overset{+}{B}$), the group is denoted by $A \supset B$ or $A \rightarrow B$ and is called material implication of B by A.

This last connection between A and B is exclusively based on their « truth values » and not on any relation between the form or content of A and B, so that it can only be established when from other sources A and B are known to be true or false. The idea is based on the general requirement for inference that false statements should not follow from true ones, and disconsiders any other feature. As an abstract game material implication is harmless enough but the real question is whether it is useful.

As a matter of fact this empty idea of implication has been invented for the examination of antinomies in mathematics i. e. for the solution of the problem of consistency in that science. Antinomy means that two contrary statements A and A′ have been deduced from admitted premises by admitted methods, i. e. inconsistency has been established in the system S in question. The only remedy seems to be to restrict the premises in S or the methods of inference used in S or both. For this purpose we have to show that A and A′ follow from certain premises by certain methods, and that they do not follow from other premises or by other methods *irrespective of the truth value of premises and* A, A′.

Consistency is a kind of coherence displayed by a set of statements, like the axioms of Euclidean geometry, and it means that these statements and their consequences do not clash. Since

purposes, since we cannot base *our* judgment on our *ignorance.* The only possible attitude towards ignorance is to admit it as a first step to its elimination. Therefore, for us, « true to fact » necessarily means « true to F », i. e. true to our checked information about the world.

As to abstract statements about notions, our various geometries and physical theories show that the world of concepts is full of contradictory statements, so that in this sphere only the relative meaning of truth is available.

It follows that the belief that every statement is either true or false is a naive and empty conception, for, in fact, true and false without a system of reference have no precise meaning. In some cases such naive beliefs may become directly obnoxious by hiding under their cover the complexity of the real problem, in this case, that of finding effective methods to decide between true and false.

The retarding effect of such empty conceptions has been clearly shown in the case of relativity theory. The anthropomorphic picture of the universe as made up of successive moments constituted by the known *and unknown* events that happen at that moment veiled for a long time the relative nature of simultaneity. Unless we give a humanly accessible and verifyable meaning of « simultaneous events at remote parts of the universe », the conception of their simultaneity or otherwise is, at least for scientific purposes, perfectly gratuitous, empty.

The definition of simultaneity by light-waves may be imperfect, may contain doubtful assumptions, but we can work with it i. e. it leads to statements we can verify and rely upon in further investigations. Also the method of construction is frankly given with all reservations for possible errors.

The absolutist outlook in physics as well as in logic and in mathematics is distinctly theological, while the relativist outlook is human and scientific (anthropo-logical as opposed to theological).

We remark however that classical logic properly (and modestly) formulated is perfectly sound as it is based on undeniably true properties of collections of definite objects, but it does not go very far as it only deals with a restricted type of relationship between collections.

objects, (ii) consistency is perceived to be invariant under the devolving process of increasing step by step the number of collection variables. Thus ultimately both in arithmetic and in L. C. consistency is based on the same intuition, i. e. on the general properties revealed by the process of systematic (devolving) growth of collections. Therefore, within its own field, L. C. admits the principle of contradiction in the form that the corresponding $\overset{+}{S}$ does not contain statements that exclude each other; there are not contradictory or contrary statements in $\overset{+}{L}$. C. proper.

The principle of excluded middle has no clear meaning in L. C., since the frame of reference for the attribute « not-true » is in general not given unless a construction for the artificial counterparts of statements is specified. Even then, most statements will be neither true nor false for L. C. We notice finally that in a logic on a restricted field no *general* principles of identity, contradiction, and excluded middle can be formulated or made use of.

24. — *Absolute logic.* — In this section I shall formulate my attitude towards the claim of logicians and philosophers to state the principles of logic in an absolute sense without an explicitly given system of reference (of statements) in the background. However, the trend of argument of the paper is independent of the following remarks.

Is there such a thing as absolute truth? In a sense there is. I accept the verdict of our direct experiences as final, subject to reservations due to optical and other illusions. In this sense « true to fact » is tantamount to absolute truth. But our knowledge of facts is so fragmentary that relatively few statements can be directly checked by facts.

« Things do happen somehow, and we either guess how they do, or we do not ». Things certainly happen somehow but unless we *know* how, we cannot use them to check our statements. Since *we* have to separate reliable statements from unreliable ones, we can only use *our own* incomplete information available at the moment; i. e. our guess may remain a guess, neither true nor false. « Reality in toto » as an omnipotent arbiter of truth is a naive conception, perfectly useless for human

naturally depends on the extent of negation in the first denial as well as on its own frame. If « not brown » means one of the other colourings and *if we keep to the same frame*, « not not-brown » means brown. In symbols, if we negate $\bar{a}c$ in the frame c, viz. if we do not negate the frame itself, $\bar{\bar{a}}c = ac$.

Besides the negation of subject or predicate (internal negations) there is an external form of negation : « this chair is brown » is not the case. Its usual meaning is, I believe, the negation of the predicate. In this external form the true character of negation is easily recognizable. All the three forms are founded on the fact that « this chair » has been found « black », say, when it was expected to be brown. The point is that we would not have chosen « not-brown » if we had no special reason for doing so, and without a definite expectation we would simply say « this chair is black ». In science, especially, we use negation in this sense when we try to verify the consequences of a theory.

The logic of collections is very restricted in its scope, it only deals with some particular types of statements about collections and thus it is necessarily indifferent to other statements. Within its own field the statements are derived from the perception (intuition) of general properties of collections, and these statements, *if questioned*, would be declared to be true or even undeniably true.

Also, when we say that $ab + a\bar{b} > a$ is absurd, we state something definite, viz. that there is no pair of collections a and b that satisfies it, and this is true.

For the formal application of the principles of contradiction and excluded middle consider a set S of statements together with a method of deriving statements from them (inference) and let $\overset{+}{S}$ denote S enlarged by its consequences. In S a statement A can only be proved, i. e. taken for true if A is in $\overset{+}{S}$, and conversely, every statement in $\overset{+}{S}$ is taken for true. Therefore the principle of contradiction can only be reasonably applied in S if there are no contradictions, i. e. contradictory statements in $\overset{+}{S}$, in other words, if S is consistent. Consistency is a logical prior to the principle of contradiction.

In L. C. consistency is accepted on the ground that (i) the properties stated are actually possessed by collections of definite

The principles of contradiction and excluded middle (tertium non datur) viz., that a statement cannot be true and false at the same time and that a statement is either true or false, i. e. there is no third or fourth etc. possible case, involve the notions of false and of negation. In our construction of L. C. we took the realist attitude by enumerating certain properties of collections like $a + b \equiv b + a$, or the three fundamental facts for inference taken as true. The two principles emerged in facts like $a \geqslant 0$ meaning that every collection is either empty or it contains some objects and there is no third case, but « false » and « negation » were not explicitly used in the construction. It is an interesting fact that a complete logic including classical logic can be built up without explicitly using these two negative notions.

Absurdities may be said to be false, also $a + b \not\equiv b + a$ may be formally introduced as counterparts of accepted statements and they may be needed in problems. As to negation we accept its common place interpretation according to which « this chair is not brown » means « this chair has another colouring » and « not this chair is brown » means « another chair is brown », i. e. « not brown » is the disjunctive combination of colourings except brown, « not this chair » means any other chair. Thus the negation of predicate and subject involves a reference to two more or less explicitly given collections (i) a collection S of objects the subject is taken from, (ii) a collection P of attributes the predicate is taken from, in both cases, at the exclusion of the other members of the collection.

Every denial is thus a disjunctive combination of ordinary positive statements made definite by a « collection of reference » or frame like colours or chairs. Such a frame can also be detected in positive statements since « this chair is brown » really means « this chair (as distinguished from other chairs) is brown (and not red or yellow, etc.) ». For a negation, however, an explicite frame seems to be more essential, for it is this frame that determines *the extent of the negation* i. e. the full content of the statement.

This conception of negation leads to the rule that only positive statements can be denied and thus a negative statement can only be negated if it is first transformed into the positive statement of its content. Then, of course, the problem of double negation disappears in the sense that the meaning of the second negation

1. Prove that $\frac{a}{b} \rightarrow \frac{ac}{b}$, $\frac{a}{b} \rightarrow \frac{a\bar{c}}{b}$.

From $a\bar{b} = 0$ we conclude that $a\bar{b}c = 0$ and $a\bar{b}\bar{c} = 0$, and write these two equations in the form $ac.\,\bar{b} = 0$, $a\bar{c}.\,\bar{b} = 0$.

2. Prove that $\frac{a\bar{b}}{c} \longleftrightarrow \frac{a\bar{c}}{b}$.

Lefthand side means $a\bar{b}.\ \bar{c} = 0$, righthand side means $a\bar{c}.\ \bar{b} = 0$.

3. Prove that $\frac{a}{b}\ \frac{c}{d} \rightarrow \frac{ac}{bd}$, $\frac{a}{b}\ \frac{c}{d} \rightarrow \frac{a+c}{b+d}$.

From premises :

$$a\bar{b}c = 0,\quad a\bar{b}\bar{c} = 0,\quad a\bar{b}d = 0,\quad a\bar{b}\bar{d} = 0$$
$$a\bar{d}a = 0,\quad c\bar{d}\bar{a} = 0,\quad c\bar{d}b = 0,\quad c\bar{d}\bar{b} = 0.$$

Adding the first equations we get $ac(\bar{b} + \bar{d}) = 0$ i. e., $ac\bar{b}\bar{d} = 0$. Adding the last two, we get $(a + c)\,\bar{b}\bar{d} = 0$, i. e. $(a + c)\,\overline{b + d} = 0$.

We notice that in this example the premises contain no restriction on the relation between *a* and *c*, *a* and *d*, *b* and *c*, *b* and *d* in the sense that any of the twelve collections $\bar{a}c$, ac, $a\bar{c}$, ..., $b\bar{d}$, bd, $\bar{b}d$, may be empty or not independently of the others, and therefore the syllogistic formulation of the argument is replaced by what classical logicians may call immediate inferences.

23. — *Negation and principles of logic.* — The principle of identity in the logic of collections (L. C.) consists of the convention that a collection is considered as entirely and exclusively determined by its objects (supposed to be definite). Its real content is rather negative since it proposes to disconsider order, grouping, and other particularities of the totality. In mathematics we go further in the elimination of properties, for we also consider two equally numerous collections, irrespective of their objects, as identical for our purpose. Thus the principle of identity in mathematics is more abstract than the corresponding principle in L. C. On the other hand, in L. C. we do not examine precise numerical relations between collections, and, in the theory of groups we also study the structure of a collection so that in many respects mathematics is more concrete than L. C. Thus the principle of identity appears as a convention that fixes the point of view adopted in a particular research, and it seems to be based on the fact that we can keep to a definite set of conventions, a fact that is not obvious nor trivial in our constantly changing world.

$a > 0 \rightarrow ab > 0$ (partial consequence), (ii) $ab = 0$ and $a > 0$ $\rightarrow a\bar{b} > 0$, proved by the formula $a = ab + a\bar{b}$.

The four cases of obversion are (i) $a \leqq b \rightarrow a\bar{b} = 0$, (ii) $ab > 0$ $\rightarrow \overline{a\,(\bar{b})} > 0$, (iii) $ab = 0 \rightarrow a = a\bar{b}$, (iv) $a\bar{b} > 0 \rightarrow a\,\overline{(b)} > 0$.

The two cases of contraposition are (i) $a \leqq b \rightarrow a\bar{b} = 0$, (ii) $a\bar{b} > 0 \rightarrow \bar{b}a > 0$.

The final problem of traditional logic is to « prove » the immediate inferences and at present there is no uncontested way of doing it. The position seems to be this. The three laws of logic are not sufficient for the purpose since their content is rather negative and the actual fundamental principle of syllogism, the *Dictum de omni et nullo*, cannot be reduced to them. Now, all the immediate inferences of traditional logic are included in the logic of collections as developed in this chapter and in the preceding one, so that we can « prove » them by justifying our theory of inference.

Our fundamental test « less information follows from more information » (equality included for formal simplicity) and the technique of inference are based on three general properties of collections : (i) if the part exists, so does the whole, (ii) if the whole is empty, so are its parts, (iii) the part of a part is a part of the same whole, i. e. on the twofold but one-sided existential relation between parts and the whole collection and on the transitive character of the relation of a part to the whole collection. A friend of a friend may not be a friend so that friendship is not a transitive relation ; also the existential relation is two-sided. Therefore the logic of this relationship would look very different from the logic of collections.

Ultimately, therefore, the logic of collections is built on some general, uncontested properties of collections, it is in fact the characteristica of the mere idea of collection where structure is systematically ignored. The theory of inference in traditional logic is really built on the same foundation since existential implications make the extension, i. e. collection interpretation imperative. Thus our theory of inference is an improved edition of the classical theory, and we have avoided the grammatical acrobatics of the latter by the help of an adequate symbolism.

We give some examples to illustrate the direct way of arguing in L. C. (logic of collections) as opposed to syllogisms.

To follow to a certain extent these deformations of sentences in our symbolic language we complete it by crossing $=$ or $<$ or $\leq$ to indicate negation. Thus $\bar{A}$ is $a \not\leq b$, $\bar{E}$ is $ab \neq 0$, $\bar{I}$ is $ab \not> 0$, $\bar{0}$ is $a\bar{b} \not> 0$.

We notice however that we can easily transpose these formally negative statements into ordinary statements. For ex., $a \not\leq b$ is fully equivalent to $a\bar{b} = 0$ or $a = ab$ and if ab is *not* empty then $a\bar{b} > 0$, so that $\bar{A}$ is in reality $a\bar{b} > 0$, $\bar{E}$ is $ab > 0$, $\bar{I}$ is $ab = 0$, $\bar{0}$ is $a\bar{b} = 0$.

In classical logic however the relations are described in colloquial sentences, and thus logicians had to find out what kind of grammatical alterations lead to equivalent statements or to consequences. Their results conveniently tabulated form the doctrine of immediate inference [1]. For example, for opposition and subalternation they have found that

$A \rightarrow (\bar{E}),\quad (I),\ \bar{0}.$	$\bar{A} \rightarrow 0$
$E \rightarrow (\bar{A}),\quad \bar{I},\ (0).$	$\bar{E} \rightarrow I$
$I \rightarrow \bar{E}.$	$\bar{I} \rightarrow (A),\ E,\ (0)$
$0 \rightarrow \bar{A}.$	$\bar{0} \rightarrow A,\ (\bar{E}),\ (\bar{I})$

where the bracketed conclusions are not generally valid if also the empty collection is admitted. The others can be « proved » by rewriting the inference in our symbols. For instance $A \rightarrow \bar{0}$ means that $a\bar{b} = 0 \rightarrow a\bar{b} \not> 0$ i. e. $a\bar{b} = 0$, the underlying fact being that for any collection *a* the two cases $a > 0$ and $a = 0$ are mutually exclusive and one of them is always the case.

$A \rightarrow \bar{E}$ means $a\bar{b} = 0 \rightarrow ab \neq 0$, i. e. $ab > 0$ which is only true if $a > 0$. In the latter case $a\bar{b} = 0$ *and* $a > 0$ entail R_{34}, and $ab > 0$ is the only general consequence on the component parts and thus, from out actual point of view, $ab > 0$ can be considered as the full consequence of the premises although from $ab > 0$ it only follows that $a > 0$ and $a\bar{b}$ may be empty or not.

The table for simple conversion only contains two items transcribed into (i) $ab > 0 \rightarrow ba > 0$, (ii) $ab = 0 \rightarrow ba = 0$; the table of conversion by limitation reduces to (i) $a\bar{b} = 0$ and

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[1] See for instance J. Jörgensen. A treatise of formal logic. Vol. II., pp. 17-20.

$c > 0, \frac{a}{b} b/c \to \genfrac{}{}{0pt}{}{a}{=}\!\!\genfrac{}{}{0pt}{}{}{c}$ transposed form of weakened Camenes or Camestres.

$b > 0, a/b \frac{b}{c} \to \genfrac{}{}{0pt}{}{a}{=}\!\!\genfrac{}{}{0pt}{}{}{c}$

$b > 0, \frac{b}{a} \frac{b}{c} \to a \,||\, c$ Darapti.

Grammatically $ab = 0$ can be expressed in two fully equivalent ways : no *a* is *b* and no *b* is *a*. Similarly $ab > 0$: some *a* is *b*, some *b* is *a*. Since these two grammatical sentences describe the *same* logical relation, there is no reason whatever to consider Ferio, Festino, Ferison and Fresison, for example, as distinct moods. All the four moods have the same relations between a, b and b, c for their premises and lead to the same relation between *a* and *c*. Also, some of the accepted moods like Celarent and Camestres are transposed of one another, and some of the transposed moods are omitted because the conclusion cannot be formulated in the accepted sentence form so that it has *a* for its subject.

22. — *Classical theory of inference.* — In classical logic syllogisms are « proved » by « immediate inferences » so that the doctrine of these immediate inferences is the base of Aristotlelian logic. Consider the four types of statements

A : all a are b, i. e. $a \leq b$ or $a\bar{b} = 0$ or $a = ab$

E : no a is b, i. e., $ab = 0$

I : some a are b, i. e., $ab > 0$

O : some a are not b, i. e., $a\bar{b} > 0$;

call *a* the subject (S) and *b* the predicate (P), introduce the negation of S and P by no or not, call quality the positive or negative character of the statement and call quantity the « all » or « some » character. The immediate inferences are then obtained by the five operations :

I. Opposition. Change of quality (negation).

II. Subalternation : change of quantity.

III. Conversion (i) simple : exchange of P and S, (ii) by limitation : also reduce the quantity.

IV. Obversion (equipollence) : change of quality and negation of P.

V. Contraposition : change of quality, exchange of S and P and negation of subject.

R'_{34}, R''_{34}, the conclusions cease to be dichotomic. On the other hand, since $ac > 0$, we obtain also Bramantip transposed, i. e. *a* and *c* exchanged.

For another example consider $a\bar{b} > 0$ and $c\bar{b} > 0$. Decomposition leads to $a\bar{b}c + a\bar{b}\bar{c} > 0$, $ac\bar{b} = 0$, $\bar{a}c\bar{b} = 0$ and, by the first equation, the inequality reduces to $a\bar{b}\bar{c} = 0$, whose full consequences are $a\bar{b} > 0$ and $a\bar{c} > 0$. Picking out the last part of the consequence as the only new piece of information we obtain

$$(a\bar{b} > 0)\ (c\bar{b} = 0) \rightarrow a\bar{c} > 0 \text{ (Baroco)}$$

For an unsuccessful example consider $a\bar{b} = 0$ and $b\bar{c} > 0$, i. e. $\bar{a}bc = 0$, $a\bar{b}\bar{c} = 0$, $ab\bar{c} + \bar{a}b\bar{c} > 0$. This system of conditions leads to no consequence on (a, c), i. e., the full consequence is denoted by R_{1234}, R'_{∞}, R''_{5678}.

If we disregard transposed moods on the ground that our results apply to any collections *a*, *b*, *c*, and therefore also to the case where *a* is *c* and *c* is *a*, we see that in the conception of classical logic there are only four different types of possible premises and conclusions : $a\bar{b} = 0$ denoted by $\frac{a}{b}$, $b\bar{a} > 0$ denoted by $\overset{a}{\underset{b}{=}}$, $ab = 0$ denoted by a/b and $ab > 0$ denoted by $a \parallel b$. Going through the various combinations (see Dienes (3)) we readily verify that there are only five really different types of combinations leading to an admitted type of conclusion.

Primary moods.

$\frac{a}{b}\ \frac{b}{c} \rightarrow \frac{a}{c}$	Barbara
$\frac{a}{b}\ \overset{b}{\underset{c}{=}} \rightarrow \overset{a}{\underset{c}{=}}$	transposed Baroco
$\frac{a}{b}\ b/c \rightarrow a/c$	Celarent, Cesare and transposed Camestres, Camenes
$\frac{a}{b}\ c \parallel a \rightarrow c \parallel b$	transposed Disamis, Dimaris, Darri, Datisi
$a/b.\ b \parallel c \rightarrow \overset{a}{\underset{c}{=}}$	transposed Ferio, Festino, Ferison, Fresison.

Secondary moods.

$a > 0, \frac{a}{b}\ \frac{b}{c} \rightarrow a \parallel c$	weakened Barbara, transposed Bramantip.
$a > 0, \frac{a}{b}\ b/c \rightarrow a \parallel c$	weakened Celarent

a and b is as follows. We determine the parts $a\bar{b}$, ab, $\bar{a}b$ that include components of x, y, z. If there are no such parts, formulæ in x, y, z have no consequences on the relation between a and b. If $a\bar{b}$, say, contains a component part of x, y, z, and if in virtue of S this component is > 0, then $a\bar{b} > 0$. Elementary equations can only have consequences on $a\bar{b}$ if $a\bar{b}$ is a sum of component parts of x, y, z all empty in virtue of S.

21. — *Syllogisms of classical logic.* — A conjunctive (simultaneous) combination of distinct elementary equations can be condensed into one formula, for, if a and b are distinct, $(a = 0) \times (b = 0)$ and $a + b = 0$ are equivalent. A disjunctive (alternating) combination of distinct elementary inequalities can also be condensed into one formula, for $(a > 0) + (b > 0)$ and $a + b > 0$ are equivalent if a and b are distinct. For instance $(a\bar{b}c = 0)\ (a\bar{b}\bar{c} = 0)$ can be written in the simple form $a\bar{b} = 0$, and $(a\bar{b}c > 0) + (a\bar{b}\bar{c} > 0)$ in the form $a\bar{b} > 0$.

The classical (Aristotelean) theory of inference only considers premises and conclusions of the above type. To show the place of this theory in our treatment of inference consider the six simple dichotomic conditions on the relation between any two collections a and b, viz. $a\bar{b} = 0$, $a\bar{b} > 0$, $ab = 0$, $ab > 0$, $\bar{a}b = 0$, $\bar{a}b > 0$, and take such a condition on (a, b) and another on (b, c) and determine their consequences, especially on (a, c). In the latter case the problem may be called the elimination of b.

Take $a\bar{b} = 0$ and $b\bar{c} = 0$. Following the method of the previous section we split them into their component equations $a\bar{b}c = 0$, $a\bar{b}\bar{c} = 0$; $ab\bar{c} = 0$, $\bar{a}b\bar{c} = 0$, and notice that only the two middle equations combine into a pure condition on (a, c), viz. $a\bar{c} = 0$. The full consequence of $(a\bar{b} = 0)\ (b\bar{c} = 0)$ on the relations between (a, b), (a, c) and (b, c) are adequately described by R_{1234}, R'_{1234}, R''_{1234}, but R_{1234} and R''_{1234} are obviously displayed by the premises themselves so that only R'_{1234} appears as a *new* piece of information that is not contained in either of the premises taken separately. Thus we obtain

$$(a\bar{b} = 0)\ (b\bar{c} = 0) \rightarrow a\bar{c} = 0 \text{ (Barbara)};$$

i. e. in common language if a is in b and b is in c, then a is in c. We remark however that in classical logic the tacit assumption $a > 0$ is made, and in this case the full consequences are R_{34},

and the last by elementary inequalities, so that single elementary equations or inequalities are never absurd. Even simultaneous elementary equations and inequalities can only be absurd if they state at the same time that the same constituent part is 0 and > 0, for the constituent parts are all distinct.

The consequences of $f(x, y, z) = g(x, y, z)$ are then determined in the following way. We reduce f and g into sums of component parts, suppress equal component parts that occur on both sides (for $a = b + a$ is fully equivalent to $b = 0$) and equate the remaining parts separately to 0 (for, when a, b and c are distinct, $a + b = c$ is fully equivalent to $a = 0$, $b = 0$, $c = 0$). Then we determine the consequences of the elementary equations and take the common part of the relations admitted by them.

The consequences of $f(x, y, z) > g(x, y, z)$ are obtained in a similar way because $a + b > a + c$ is equivalent to $b > c$, and $a + b > 0$ is equivalent to $a > 0$ *or* $b > 0$. The only difference is that at the end instead of taking the common part we pool together the admitted relations.

If we have a set of formulæ composed of conjunctive and disjunctive combinations of other formulæ, we reduce the set to a sum of direct products Σ ABC..., solve A, B, C, ... separately, take the common solutions of the factors and pool the information contained in the terms.

We notice that absurdities in a set S do not necessarily make S useless for inference. In fact with regard to the relations between (x, y), (x, z), (y, z) the inference value of $xyz > 0$ *and* $xyz = 0$ on their component parts is $xy > 0$, $xz > 0$, $yz > 0$, for $xyz = 0$ entails no consequence on these component parts. No information, if a factor, behaves like 1, i. e. can be suppressed. In general $(f > 0) + (f = 0)$ is such a factor with the difference that this sum can also be suppressed when it figures as a term whereas $xyz = 0$ may occur together with $xy\bar{z} = 0$, say, and then they entail $xy = 0$. There are cases, however, where $xyz = 0$ does contain information; for instance if $a = xy$ and $b = z$, $xyz = 0$ means $ab = 0$. In such a case the absurd combination becomes useless for it entails both R_{3478} *and* R_{1256}, i. e. together they admit no possible relation between xy and z.

The general procedure to find the consequences of a set of formulæ S in x, y, z on the relation between two given collections

Take xyz = 0, whose effect on the relation between a = x and b = yz, say, is that ab = 0 and a$\bar{b}$, $\bar{a}$b may be empty or not, so that (a, b) cannot be in relations r_5, r_6, r_7, r_8 and may be in r_1, r_2, r_3, r_4, a fact we shall denote by R_{1234}. Conversely if x and yz cannot be in r_5, r_6, r_7, r_8, then xyz = 0.

Specially interesting are the consequences on x, y, z themselves. For ex. the condition xyz > 0 entails that x and y are in R_{3478}, x and z in R'_{3478} and y and z in R''_{3478} where the dash and double dash indicate the relationship between x and z, and y and z respectively.

The inequality $xy\bar{z} > 0$ entails R_{3478}, R'_{5678}, R''_{5678} ; $x\bar{y}\bar{z} > 0$ entails R_{5678}, R'_{5678}, $R''_{12345678} \equiv R''_\infty$ (no restriction on the relation between y and z). The consequences of $xyz > 0$ *and* $xy\bar{z} > 0$ on the relation between (x, y), (x, z) and (y, z) are R_{3478}, R'_{78}, R''_{78}, the common part of the separate consequences.

It is an interesting fact that single elementary equations like xyz = 0 entail no consequences on (x, y), (x, z), (y, z), a result indicated by R_∞, R'_∞, R''_∞. Even two or three such equations ex. $\bar{x}\bar{y}z = 0$, *and* $\bar{x}y\bar{z} = 0$ *and* $\bar{x}\ z\ \bar{y} = 0$ may have no consequences of the type in view. There is only one group of four elementary equations viz. $x\bar{y}\bar{z} = 0$, $\bar{x}y\bar{z} = 0$, $\bar{x}\bar{y}z = 0$, $xyz = 0$ that yields no consequences on (x, y), (x, z), (y, z). The rule is that unless two simple products have *two* common factors, the corresponding simultaneous equations yield no consequences of the required type, and if they have two common factors, the equations lead to a fourfold (dichotomic) consequence on the common factors obtained by equating the common (double) factor to 0, and yield no consequences on the other two pairs. For example $x\ \bar{y}\ \bar{z} = 0$ and $x\ \bar{y}\ z = 0$ give R_{1234}, R'_∞, R''_∞, best represented by the formula $x\bar{y} = 0$. This result can be formally obtained by adding the two parts and dividing by $z + \bar{z}$, which is legitimate since $a(b + \bar{b}) = 0$ and $a = 0$ imply each other, and also $a + b = 0$ and $(a = 0, b = 0)$ imply each other.

For an inequality like $xy\bar{z} > 0$ the formal rule is to write $xy > 0$, $x\bar{z} > 0$, $y\bar{z} > 0$ which, by Table II, display the admitted and excluded relations R_{3478}, R'_{5678}, R''_{5678}. Notice however that $x\ \bar{y}\ \bar{z} > 0$ only gives $x\bar{y} > 0$ and $x\bar{z} > 0$, i. e. R_{5678}, R'_{5678}, R''_∞. The two inequalities $xy\bar{z} > 0$ and $x\bar{y}\bar{z} > 0$ give R_{78}. R'_{5678}, R''_{5678}.

The first relation cannot be excluded by elementary equations

fication, i. e. conclusions are drawn from premises that may turn out to be untenable.

Another feature of inference is that the conclusion is drawn on general grounds (if the part exists, so does the whole) and not by actually verifying that $a\bar{b} > 0$ has less roots than $a > 0$ (as a formula in a, b). For instance the passage from the premises $a < b$ and $b < c$ to the conclusion $a < c$ (Barbara) is justified by noticing that « a part of a part is itself a part » (of the same whole). Therefore the technique of inference must be based on the import and structure of statements and not on their truth, and we shall have to justify this technique by an appeal to some simple facts or « principles ».

We start with some examples. Two collections x and y are subjected to the condition $x < y$ and only to this condition viz. that the first is a part of the second. What can we infer from this condition about the relation between two collections a and b? Nothing, of course, unless x or y enter into their composition. Therefore we restrict a and b to collections formed by parts of x and y and by x and y themselves, and to have a representative example we add a third variable z, i. e. we restrict a and b to the constituent parts xyz, $xy\bar{z}$, $x\bar{y}z$, $x\bar{y}\bar{z}$, $\bar{x}yz$, $\bar{x}y\bar{z}$, $\bar{x}\bar{y}z$ and their sums.

If $a = x\bar{y}$ and $b = y$, we can infer from $x < y$ that $a = 0$, $b > 0$, i. e. that a and b are in relation r_2. Conversely if $x\bar{y}$ and y are in the relation r_2, then $x < y$. We also notice that the parts x and z may be in any of the eight relations and thus from $x < y$ no consequence follows on the relation between x and z.

Take now the condition $x = y$ which reduces the effective (i. e. not necessarily 0) constituent parts to xz, $x\bar{z}$, $\bar{x}z$ which may be empty or not. If $a = xz$ and $b = xy$, the three constituent parts of a and b are $ab = xz$, $a\bar{b} = 0$, $\bar{a}b = \bar{z}x$. Therefore a and b can be in any of the first four relations but not in any of the last four.

For a more systematic survey of possible inferences we shall now examine the simplest formulæ, viz. equations and inequalities between the constituent parts, and afterwards we shall split formulæ like $x = y$ into a sum of simple formulæ like $x\bar{y} = 0$ *and* $\bar{x}y = 0$. As the constituent parts are distinct, no equation or inequality is possible between them unless one of them is 0.

CHAPTER V

INFERENCE

20. — *The idea of inference.* — From the point of view of information about relations between collections the rules of manipulation like $a + b = b + a$ are mere identities; but we needed them in solving sets of formulæ, i. e. in obtaining the information contained in the set. For this work it was essential that in the transformation (reduction) of our formulæ we should not lose or gain roots, and therefore we only used identities. For ex. we replaced $A(B + C)$ by $AB + AC$ on the ground that both sets have the same roots.

There are cases, however, when we seem to be satisfied if we only know a *part* of the information hidden in a set of formulæ in which case we say that we *draw a conclusion* from the set. For instance, it « follows » from $a\bar{b} > 0$ that $a > 0$, for $a\bar{b}$ is a part of *a* and when the part exists, so does the whole. In such cases the conclusion formula admits more roots, i. e., more possible relations than the premise formula, and thus the conclusion contains less information than the premise. Also, « Socrates is mortal » contains less information than « all men are mortal » or « Socrates is a man ». The underlying principle of inference seems to be that *less information follows from more information.*

An essential feature of this process is illustrated by the fact that « if $a\bar{b} > 0$, then $a > 0$ » does not require actual specification of *a* and *b* to make $a\bar{b} > 0$ true. We only say that whenever $a\bar{b}$ happens to contain some objects, then also *a* contains some. This hypothetical character of inference is important in scientific theories where consequences are subjected to experimental veri-

Now, by Rule I, we can solve any set of formulæ if (i) we solve every formula that occurs in the set, (ii) we form the sets of solutions corresponding to the successive steps in the composition of the given set of formulæ. In the previous work we have shown how to reduce functions in a, b to bilinear functions and how to solve any formula in bilinear functions of a, b, i. e. were are in the position to carry through (i) in the case of two unknowns. In a, b, c the functions reduce to trilinear functions and any formula in those can be dealt with directly in a more or less systematic way.

As to (ii), we either follow the composition of the given set of formulæ or first reduce the set to a sum of formula-products by considering $\vee$ as a $+$ and & as a multiplication, a process which is readily justified, then for every term we form the product of the collections of roots corresponding to the factors and finally we sum the common parts so obtained. Whichever way we proceed we can slightly simplify our work by rejecting every product that contains an absurd factor.

and, apart from the genuine bilinear functions

$$(L_2') \qquad a\bar{b},\ ab,\ \bar{a}b \text{ and } a\bar{b} + \bar{a}b,$$

they are identical with one of the linear functions 0, a, b, a + b.

The *bilinear equations and inequalities* are obtained by writing $=$, $>$ or $<$ between any two linear or bilinear functions. For ex. $ab = 0$ has four roots r_1, r_2, r_4, r_5 and the other four relations are the roots of $ab > 0$. Such a formula or set of formulæ, in any number of variables, admitting half of the possible roots and rejecting the other half is called *dichotomic*, important in syllogistic reasoning. Any of the three constituent parts $a\bar{b}$, ab and $\bar{a}b$ put > 0 or $= 0$ produces a dichotomic formula and Table II shows their roots.

Since $a\bar{b}$, ab and $\bar{a}b$ are distinct and $\bar{a}b + a\bar{b}$ is distinct from ab, Rules III and IV settle equations and inequalities between genuine bilinear functions (L_2'). Those between linear and our bilinear functions can be dealt with in the same way if we substitute $a\bar{b} + ab$ and $ab + \bar{a}b$ for *a* and *b* respectively. For instance, $a\bar{b} = a$, i. e. $a\bar{b} = a\bar{b} + ab$ is equivalent to $ab = 0$ and thus it is a dichotomic equation. Also, $a\bar{b} > a$ is absurd, $a\bar{b} < a$, i. e., $a\bar{b} < a\bar{b} + ab$ is, equivalent to $ab > 0$, and thus $a\bar{b} < a$ is a dichotomic inequality.

A bilinear function may also contain terms like $\overline{a + b}\,x$, $\overline{\bar{a} + b}\,x$, $\overline{\bar{a} + \bar{b}}\,x$, where x is either *a* or *b*, but by the rules of manipulation (9) — (11) they transform into threefold products containing two *a* factors (barred or not) or two *b* factors (barred or not) and thus they reduce to 0, a, b or to one of the three constituent parts of *a* and *b*.

The same remark applies to every term of a trilinear or higher function. We notice that terms like $\overline{a\,b}\,x$, $\overline{\bar{a}\,b}\,x$, $\overline{\bar{a}\,\bar{b}}\,x$, x is *a* or *b*, yield 0 or a constituent part or sums of these parts. The two formal operations, multiplication and barring, cannot divide a + b into any other parts than 0, $a\bar{b}$, ab, $\bar{a}b$ and their sums, and addition does not divide. This fact is reflected in our formalism by the *reduction theorem : Every function in a, b is identical with a simple bilinear function in a, b.* In three variables : *Every function in a, b, c is identical with a simple trilinear function in a, b, c.* And so on.

Both types of combinations of identities (absurdities) are themselves identities (absurdities).

Rule II. The roots of $f(a, b, c) = 0$ and $f > 0$ are complementary, i. e. if α and β are the collections of their respective roots, α and β are distinct and $\alpha + \beta = R$.

This follows from the fact that, by its construction, f denotes a collection x, viz. the empty one or a part of $a + b + c$, and $x = 0$ and $x > 0$ exclude each other and one of them is always the case. Parts of $a + b + c$ will be denoted by x, y, ...

Rule III. $x + y = x$ has the same roots as $y = 0 \lor y < x$. In such a case the two sets of formulæ are said to be *equivalent*.

Rule IV. If x and y are distinct, i. e. if $xy = 0$, $x + y = x$ is equivalent to $y = 0$.

Rule V. If x and y are distinct, r_1, viz. $a = 0$, $b = 0$, $c = 0$, is the only root of $x = y$ and both $x < y$ and $x > y$ are absurd.

Rule VI. Every equation $f = f'$ is satisfied by r_1 and no inequality is satisfied by r_1, for in every term of f and f' there is a non-barred a or b or c. It follows that an equation is never absurd and an inequality never degenerates into an identity.

We shall apply these remarks to formulæ in a, b. The only linear functions in a, b are 0, a, b and $a + b$ and correspondingly *linear equations* in a, b are of the types

$$(E_1) \quad (1)\ a = 0, \quad (2)\ a + b = 0, \quad (3)\ a = a, \quad (4)\ a = b, \quad (5)\ a + b = a,$$

and as is readily verified by Table II, their roots are

r_1, r_2; r_4; all the eight relations; $r_1\ r_3$; r_1, r_5, r_7 respectively.

Linear inequalities are of the types

$$(I_1) \quad (1)\ a > 0, \quad (2)\ a + b > 0, \quad (3)\ a > a, \quad (4)\ a > b, \quad (5)\ a > a + b, \quad (6)\ a < a + b.$$

Roots of (1) and (2) are given by Rule II, (3) and (6) are absurd, the roots of (4) are r_5 and r_7, and those of (5) are $r_2, r_3, r_4, r_6, r_7, r_8$.

A complete list of simple bilinear functions in a, b is

$$(L_2) \quad aa,\ a\bar{a},\ ab,\ a\bar{b},\ b\bar{a},\ bb,\ b\bar{b},\ aa + bb,\ ab + a\bar{b},\ ab + \bar{a}b,\ a\bar{b} + \bar{a}b,$$

a, which implies that *b* is not empty. When $a = 0$, $a < b$ reduces to the simple existential statement $b > 0$. We shall interpret $a < b$ as a restriction (requirement) on the possible relations between *a* and *b*, in which case it admits r_2, r_3 and rejects the other relations, so that r_2 and r_3 may be called its roots. The general form of admissible inequalities will be $f_1(a, b, c, \ldots) > f_2(a, b, c, \ldots)$ where f_1 and f_2 are admissible functions.

We notice that r_1, i. e., $(a = 0, b = 0, c = 0, \ldots)$ cannot be a root of an inequality, for in this case both sides reduce to 0. Thus an inequality never degenerates into an identity but, on the other hand, it may have no root at all, e. g., $ab + a\bar{b} > a$, i. e. it may be an absurdity.

Identities and absurdities give no information about relations between collections, and thus they are not proper statements. They result from the conventions used in the formal construction of our logic reflecting its structure. Their admittance considerably simplifies our machinery.

The equation $ab = 0$, whose roots are r_1, r_2, r_3 and r_4, is also denoted by $a \leq b$ and is called *inclusion*, for, in this case, *a* and *b* are either identical or $a < b$.

19. — *The technique of formulae. Reduction of functions* [1]. — In this section we shall show how to solve sets of equations and inequalities in collection variables. First we formulate some more or less general rules that immediately follow from the definitions, and, for simplicity, we restrict the number of unknowns to three, a, b, c, and denote by $R(r_1, r_2, \ldots, r_{128})$ the complete collection of the 128 possible relations between them. The collection of roots of a formula $A(a, b, c)$ will be denoted by α. If $\alpha = R$, A is an identity; if α is empty, A is absurd.

Rule I. If α and β are the collections of the roots of A and B respectively, the collection of the roots of their conjunctive combination (A, B) is $\alpha\beta$, and that of the roots of their disjunctive combination $A + B$ is $\alpha + \beta$.

In particular an absurd formula in a conjunctive combination makes the combination absurd, but a disjunctive combination is only absurd when all its constituent formulae are absurd.

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(1) This section is not strictly necessary for the understanding of the trend of argument.

formally obtained by multiplying $a\bar{b}$, ab, $\bar{a}b$ *and* $\bar{a}\bar{b}$ into $c + \bar{c}$ and separating the terms. The formal product $\bar{a}\bar{b}\bar{c}$ is rejected as meaningless in itself. The eight relations are replaced by $2^7 = 128$ different relations between *a*, *b*, *c*, and these relations are mutually exclusive, and one of them is always the case. Extension to four or more collections is immediate.

There are, however, cases where our information about the relation between collections is not complete; for instance when we only know that certain relations are excluded. For two collections there are $2^8 = 256$ different informations (no information also included) of this type considered in the logic of collections. As every relation is described by a conjunctive combination of three simple existential sentences, every information can be described as a disjunctive combination of conjunctive groups of existential sentences. These combinations will be called *statements* on relations between collections, or statements for short.

To simplify the description and handling of statements we slightly generalise the two types of existential sentences into *equations* and *inequalities*, also called *formulae*. As an introduction to these ideas we notice that, unlike $a + b \equiv b + a$, $ab \equiv a\bar{b}$ is not a true identity, for it is *not* satisfied by every pair of collections. It is satisfied however if (and only if) *a* and *b* are in the relation r_1. Therefore we re-write it in the form $ab = a\bar{b}$ and consider it as a restriction (or requirement) on the possible relations between *a* and *b*. We also say that r_1 is a (relation —) root of that equation.

The roots of $ab + \bar{a}b = 0$ are r_1 and r_5, and $b = 0$ considered as an equation in *b and a*, has the same roots. The general form of *admissible equations* will be $f_1(a, b, c, \ldots) = f_2(a, b, c, \ldots)$ where f_1 and f_2 are admissible functions. We notice that in f_1 and in f_2 every term contains a non-barred symbol and thus the equation is certainly satisfied by $a = 0, b = 0, c = 0, \ldots$ i. e. every admissible equation or set of such equations admits the trivial solution r_1. From this relational point of view an identity is an equation satisfied by all possible relations so that in this connection $\equiv$ may be replaced by $=$.

The inequality $a < b$ will mean that *a* is a genuine part of *b*, or, more precisely (i) *b* contains every object of *a* when the latter is not empty, (ii) *b* also contains at least one object that is not in

we supplement them by the meaningless operational rule $\bar{\bar{a}} \equiv a$.

A finite succession of additions, multiplications and barrings performed on a, $\bar a$, b, $\bar b$ in a given order is denoted by f(a, b) and is called a *form* in *a* and *b* or a *function* of *a* and *b*. This definition readily extends to more than two « variables ».

18. — *Relations, statements, formulæ.* — In this general inquiry into the nature of the notion of collection we not only disregard the order and grouping of objects but we also ignore their exact numbers, i. e. we restrict ourselves to two types of information about collections, viz. « *a* is empty » and « *a* contains some objects ». These simple existential sentences are denoted by $a = 0$ and $a > 0$ respectively. For any collection *a* the attributes $= 0$ and > 0 exclude each other and one of them is always possessed by *a*, i. e. they are mutually exclusive and alternating properties of collections.

We shall also consider disjunctive combinations like « either $\bar a b > 0$ or $a\bar b = 0$ » denoted by the sign $\vee$ (vel) or $+$ between the sentences, and conjunctive combinations like « $ab = 0$ and $\bar a b = 0$ » denoted by a comma, & or $\times$ between the sentences.

As we have agreed to look at collections from the point of view of existence only, the possible relations between *a* and *b* will be restricted to the existence or non-existence of *a*, *b* and their three constituent parts $a\bar b$, ab and $\bar a b$. Thus we obtain

TABLE II

(r_1)	$a\bar b = 0,\ ab = 0,\ \bar a b = 0$	$a = 0,\ b = 0$
(r_2)	$a\bar b = 0,\ ab = 0,\ \bar a b > 0$	$a = 0,\ b > 0$
(r_3)	$a\bar b = 0,\ ab > 0,\ \bar a b = 0$	$a \equiv b > 0$
(r_4)	$a\bar b = 0,\ ab > 0,\ \bar a b > 0$	$a > 0,\ b > 0$ and a is part of b
(r_5)	$a\bar b > 0,\ ab = 0,\ \bar a b = 0$	$a > 0,\ b = 0$
(r_6)	$a\bar b > 0,\ ab = 0,\ \bar a b > 0$	$a > 0,\ b > 0$, and a and b are distinct
(r_7)	$a\bar b > 0,\ ab > 0,\ \bar a b = 0$	$a > 0,\ b > 0$, and b is part of a
(r_8)	$a\bar b > 0,\ ab > 0,\ \bar a b > 0$	$a > 0,\ b > 0$, and a and b overlap.

For any given collections *a* and *b*, these eight relations are mutually exclusive and one of them is always the case.

In case of three collections *a*, *b*, *c* we have $7 = 2^3 - 1$ constituent parts

$$a\bar b c,\ a\bar b\bar c,\ abc,\ ab\bar c,\ \bar a bc,\ \bar a b\bar c,\ \bar a\bar b c$$

Table I

$$
\begin{array}{rll}
(1) & a + b \equiv b + a & \multirow{2}{*}{\Big\} \text{ commutative rule}} \\
(2) & ab \equiv ba & \\
(3) & (a + b) + c \equiv a + (b + c) \equiv a + b + c & \multirow{2}{*}{\Big\} \text{ associative rule}} \\
(4) & (ab)\,c \equiv a(bc) \equiv abc & \\
(5) & a(b + c) \equiv ab + ac & \multirow{2}{*}{\Big\} \text{ distributive rule}} \\
(6) & a + bc \equiv (a + b)\,(a + c) & \\
(7) & a + a \equiv a & \multirow{2}{*}{\Big\} \text{ rules of tautology}} \\
(8) & aa \equiv a & \\
(9) & \overline{a + b}\,c \equiv \bar{a}\,\bar{b}\,c & \\
(10) & \overline{\bar{a} + b}\,c \equiv a\,\bar{b}\,c & \Bigg\} \text{ rules for barred sums} \\
(11) & \overline{\bar{a} + \bar{b}}\,c \equiv abc & \\
(12) & \overline{ab}\,c \equiv (\bar{a} + \bar{b})c & \\
(13) & \overline{\bar{a}b}\,c \equiv (a + \bar{b})c & \Bigg\} \text{ rules for barred products} \\
(14) & \bar{a}\,\bar{b}\,c \equiv \overline{a + b}\,c & \equiv (9)
\end{array}
$$

Rules (1) — (6) result from our convention to disconsider order and grouping within collections, i. e. from our principle of identity. Rules (7) and (8) result from the convention to take common elements once only. Rules (9) — (14) are readily verified if $\bar{a}\ \bar{b}\ c$ is interpreted in the usual way as meaning the part of *c* that is not in *a* and not in *b*. In the working of these last rules *c* plays no active qart and thus for the formal work we might formulate the rules for barred symbols together with direct, non-barred symbols as in ordinary symbolic logic. In particular it is readily seen that

$$\overline{\bar{a} + b}\ c \equiv \overline{b + \bar{a}}\ c, \quad \overline{\bar{a} + \bar{b}}\ c \equiv \overline{\bar{b} + \bar{a}}\ c, \quad \overline{\bar{a}\,\bar{b}}\ c \equiv \overline{\bar{b}\,\bar{a}}\ c,$$

so that the commutative and associative rules and similarly the distributive rule hold also for barred symbols. For the rules of tautology we only have to remark that

$$(\bar{a} + \bar{a})c \equiv \bar{a}c + \bar{a}c \equiv \bar{a}\ c$$

and that $\bar{a}\ \bar{a}\ c$ means the part of *c* that is not in *a* and not in *a*, i. e. it means $\bar{a}c$. We also notice that in case of repeated bars obtained in a formal application of rules (9) or (12) we get the right results if we replace $\bar{\bar{a}}$ by a. Therefore in our formal work we can ignore rules (10), (11), (13) and (14) and we can use all the other rules for direct or barred symbols indiscriminately if

tive character of inequalities which, in its turn, for integers at least, is based on the ultimate « fact » that the succession of integers produces numbers larger than all the previous numbers. Also when we prove that the algebra of fractions includes that of integers, we rather observe a fact than apply moods of syllogism (that may be inherent in it).

Therefore we shall first sketch the logic of collections in a form closely related to our mathematical way of thinking and then we shall determine the place of syllogistic reasoning in this scheme. In this way we shall be able to compare mathematical thought with the special syllogistic way of thinking as well as with Russell's general notion of implication.

17. — *Sums, products, functions.* — « Collection » is considered as a primitive notion but we qualify it by the convention that the order or grouping of its objects is disconsidered so that a collection is entirely and only determined by its objects. This is the principle of identity in the logic of collections. Rigid objects arranged in various orders and groups are the standard example for identical collections. This logic is thus based on the identity of individual objects. If a and b denote identical collections, we put $\mathrm{a} \equiv \mathrm{b}$ and call it an *identity*.

Since we disregard grouping of the objects in the same collection, the only operations that remain are joining new objects to it or taking away some of its objects. As the added or suppressed objects can be considered as collections and they may belong to other collections, we introduce (i) the *sum* $\mathrm{a} + \mathrm{b}$ (disjunctive combination) to denote the united collection, common objects taken once only, (ii) the *product* ab (also written $\mathrm{a} \times \mathrm{b}$) to denote the collection of the common objects, (iii) the *product* $\mathrm{a}\overline{\mathrm{b}}$ or $\overline{\mathrm{b}}\mathrm{a}$ (also written $\mathrm{a} \times \overline{\mathrm{b}}$) to denote the objects of a that are not in b. If a is empty, so is ab or $\mathrm{a}\overline{\mathrm{b}}$, and, as there is no a to restrict b, $\overline{\mathrm{a}}\mathrm{b}$ is the same as b.

The usual interpretation of $\overline{\mathrm{a}}$ as a collective symbol for everything outside a is not admissible here, for $\overline{\mathrm{a}}$ does not denote a collection. Therefore we do not attribute a meaning to isolated barred symbols though otherwise we do not restrict their use.

The formal rules of manipulation for sums and products are given in

CHAPTER IV

RELATION BETWEEN COLLECTIONS

16. — In the previous sections we have sketched the construction of arithmetic in the form of the algebra of positive and negative numbers, i. e. in the form of a closed system of symbolism developed for the exact presentation of certain fundamental experiences and now we shall turn our attention to the logical problems involved in this work. In particular, we shall discuss the problem of consistency of our structure and the definite character of arithmetical problems.

Since in all its successive stages the construction actually was a continued extension of the algebra of integers, the connection between arithmetic and its ultimate foundation does not only need special consideration, but the understanding of this connection is practically the only way to throw some light on our fundamental problems. In other words, since a kind of derivation of statements from other statements is inherent in mathematical thought, we shall have to extricate the threads and patterns of this derivation and to compare them with those of existing theories and techniques of inference.

Now arithmetic is based on the process of counting the objects of collections, and in its construction we certainly draw conclusions from premises as understood by common sense and in ordinary logic. In mathematics however the formal machinery of syllogistic reasoning is usually replaced by the very formalism of mathematics as determined by the nature of its own problems and their interdependence. For instance, when we want to prove that $a < c$ follows from $a < b$ and $b < c$, we appeal to the transi-

The rule, — times — is +, is derived from the same interpretation. If I have to add $2(x - 3)$ to S (my fortune, say), and in certain cases I may only know the value of x later, I can split the operation into two by saying that « add 2x » but also « subtract 2.3. » If I have to subtract $2(x - 3)$ from S, I « subtract 2 x » but I also « add 2.3 », for otherwise I would subtract too much. We mechanise this procedure by putting

$$-2(x-3) = -2x + 2.3, \quad -a(b-c) = -ab + ac.$$

The formal construction starts from the condition

$$\bar{a} + a = 0,$$

and from the assumption that all rules of manipulation apply. Thus

$$(1) \qquad \bar{a} + a = 0 \quad \text{and} \quad \bar{b} + b = 0$$

lead straight to

$$\bar{a} + \bar{b} + (a + b) = 0$$

showing that $\bar{a} + \bar{b}$ satisfies $\bar{x} + (a + b) = 0$, i. e.

$$\bar{a} + \bar{b} = \overline{a + b}.$$

Also multiplying by b and a respectively and adding we get from (1)

$$(2) \qquad \bar{a}\,b + a\,\bar{b} + 2\,a\,b = 0$$

and multiplying the equations (1) we get

$$(\bar{a} + a)(\bar{b} + b) = 0, \text{ i. e., } \bar{a}\,\bar{b} + a\,\bar{b} + \bar{a}\,b + a\,b = 0.$$

If we add ab to both sides and take account of (2), we obtain

$$\bar{a}\,\bar{b} = ab.$$

We now verify that for sums and products of barred or non-barred numbers all the rules of manipulation hold, o acquiring the additional set of meanings $\bar{a} - \bar{b}$, and that the old algebra is contained in the new. Also inverse operations are successfully defined as usual. An order of magnitude is ascribed to the whole range of positive and negative rational numbers by putting

$$\bar{a} < 0 \text{ and } \bar{a} < \bar{b} \text{ if } a > b.$$

even when N is not divisible by n, (ii) kn and kN determine *equal* symbols, i. e. if y satisfies

$$kn.\ y = kN,$$

we say that $y = x$. (iii) As an internal requirement we want to preserve the simple rules of manipulation — if possible.

Now, using these rules of manipulation, from $n_1x = N$ and $n_2y = N_2$ we readily get $n_1n_2(xy) = N_1N_2$ and $n_1n_2(x + y) = n_2N_1 + n_1N_2$, leading to the definitions of product and sum.

Now, we verify that (i) the simple rules of manipulation hold for sums and products so defined, (ii) for fractions with the denominator 1 the formalism reduces ot the algebra of integers where under every integer we write 1. Thus we prove that the algebra of fractions is regular and that it contains the algebra of integers, two rather fortunate facts if we realise that the symbolism has been suggested by our experiences in comparing lengths.

Zero, 0, is introduced as meaning any of the symbols $\frac{0}{1}, \frac{0}{2}, \ldots, \frac{0}{n}, \ldots$ and its usual properties in sums and products are easily established. An order of magnitude is first defined for fractions with the same denominator

$$\frac{N_1}{n} > \frac{N_2}{n} \text{ if } N_1 > N_2$$

and then extended to any fractions

$$\frac{N_1}{n_1} > \frac{N_2}{n_2} \text{ if } n_2N_1 > n_1N_2.$$

The usual definition of inverse operations complete the picture.

15. — *Negative numbers.* — If we want to describe change in length, as for instance in a thermometer, besides a unit of length we also need a starting point (initial length) and a specification of the direction of the change (increase or decrease). If the degrees above *o* are denoted by ordinary integers and fractions, those below *o* will have to be distinguished by a bar before or above them. The characteristic property of these barred numbers $\bar{a}$ or $-a$ is that an increase a of length makes them the *o* number. The typical example for such an increase and decrease is that of our fortune expressed in money. In the usual notation $3 - 5$ is interpreted as 3 to be added to my fortune and 5 to be subtracted. This amounts to the same as « 2 to be subtracted » which is transcribed by the equation $3 - 5 = -2$.

λ and μ, say, L_1 and L_2 would appear as divided into parts containing the same number of λ and μ lengths respectively, but the total length of a part in L_2 would be really different from that of a part in L_2 and thus the numerical sum $N_1 + N_2$ of parts in L_1 and parts in L_2 would not correspond to their united length.

Since any two fractions $\frac{N_1}{n_1}$, $\frac{N_2}{n_2}$ are equal to $\frac{n_2 N_1}{n_2 n_1}$ and $\frac{n_1 N_2}{n_1 n_2}$ respectively, their sum is defined by the formula

$$\frac{N_1}{n_1} + \frac{N_2}{n_2} = \frac{n_2 N_1 + n_1 N_2}{n_1 n_2}.$$

To show how deeply the construction of fractions is influenced by practical requirements we notice that formally the simplest and the most natural rule for addition would be

$$\frac{N_1}{n_1} \oplus \frac{N_2}{n_2} = \frac{N_1 + N_2}{n_1 + n_2}$$

since, in the same unit, this operation actually describes the passage from the ratios $\frac{L_1}{l_1}$ and $\frac{L_2}{l_2}$ to the ratio of $L_1 + L_2$ to $l_1 + l_2$. Also if we change the unit in the same way in both fractions, we obtain

$$\frac{kN_1 + kN_2}{kn_1 + kn_2} = \frac{N_1 + N_2}{n_1 + n_2}$$

Moreover, the operation is associative and commutative. However, under the pressure of practical considerations, we have accepted that

$$\frac{N}{n} = \frac{kN}{kn},$$

and thus we would expect that in any operation these equal quantities can replace each other, and this is not the case here since

$$\frac{kN_1}{kn_1} \oplus \frac{N_2}{n_2} = \frac{kN_1 + N_2}{kn_1 + n_2} \neq \frac{N_1 + N_2}{n_1 + n_2}.$$

Thus the circular sum is not a proper *ratio*-oper-ation.

The formal construction of the algebra of fractions is based on the requirements : (i) symbols x are needed such that

$$n\,x = N$$

the succession of integers, and from that source definiteness is obtained by a definite system of transmission.

14. — *Fractions.* — Exact science is based on measurement. To measure the length L we take a smaller length l, verify that L contains l exactly m times and then we declare m to be the measure of L in terms of l. If we cannot cover L by an exact number of l's, we cannot measure L by l. Since in practice we have to compare any two lengths, in the latter case we measure both L and l in a suitable small unit λ so that $\mathrm{L} = \mathrm{N}\lambda$, $\mathrm{l} = \mathrm{n}\lambda$ and we say that the *ratio* of L to l is as N to n. We notice that taking λ sufficiently small, according to the precision of our measuring instrument, we can always determine N and n for any pair of actually given lengths to the given approximation.

The measurement of areas (volumes) by smaller areas (volumes) not being practicable, we reduce it to measurement of lengths. Also the comparison of the number of ticks of two time pieces between two given events, the comparison of weights etc., lead to ratios.

The ratio of two lengths is determined by the two objective lengths, and the two numbers only describe the ratio in terms of a specified unit. Therefore for two given lengths the ratio is considered as the same whatever units (same for both) are used in the measurements involved. Thus if $\mathrm{L} = \mathrm{N}\lambda$, $\mathrm{l} = \mathrm{n}\lambda$, and $\lambda = \mathrm{m}\mu$, we say that the ratio of L to l is equally described by $\frac{\mathrm{N}}{\mathrm{n}}$ and by $\frac{\mathrm{mN}}{\mathrm{mn}}$, where m is any integer different from o. This leads to the definition of *equality* for fractions

$$\frac{\mathrm{N}}{\mathrm{n}} = \frac{\mathrm{mN}}{\mathrm{mn}}.$$

If $\mathrm{L}_1 = \mathrm{N}_1\lambda$, $\mathrm{L}_2 = \mathrm{N}_2\lambda$ and L_2 continues L_1, their combined length is $(\mathrm{N}_1 + \mathrm{N}_2)\lambda$ and thus the ratio of this combined length to l is $\frac{\mathrm{N}_1 + \mathrm{N}_2}{\mathrm{n}}$, leading to the definition of sums of fractions of the same denominator

$$\frac{\mathrm{N}_1}{\mathrm{n}} + \frac{\mathrm{N}_2}{\mathrm{n}} = \frac{\mathrm{N}_1 + \mathrm{N}_2}{\mathrm{n}}.$$

We have to remember however that this formula implies that the same unit is used in both fractions. If the units were different,

any number o times we may say that we do not take it at all. For our technique however we need a more formal definition of sums and products containing o.

12. — *Division.* — If we group the objects of C into collections of the same number b of objects, in many cases there will be some objects less than b in number left ungrouped. For given a and b, the technique of division leads to the « calculation » of the number of groups and the number of ungrouped objects, without carrying out the actual grouping and without counting the ungrouped objects. At every step this technique consists of trials on small collections and the aim of the technique is to reduce the number of trials to the minimum by adopting a systematic way of procedure. The justification of this method is based again on the structure of decimal notation and on simple cases of division actually verified.

As to division by o, we can give no satisfactory definition and thus this operation is banished from arithmetic.

If, in the division of c into groups of b objects, no objects are left ungrouped and the number of groups is a, we have $c = ab$, i. e. c appears as a product of two smaller numbers. It also happens e. g. if $c = 2, 3, 5, 7, 11, 13$ etc. that c cannot be so divided into groups without some objects remaining ungrouped. Such numbers we qualify as *prime* numbers. If both a and b are primes, we say that we split c into its prime factors. If a (or b) is not prime we split a (and b) into factors, and, since at every step we obtain smaller integers, in a finite number of steps we arrive at a product form of c where all the factors are primes. In this way we « prove » that every number is either prime or the product of a finite number of prime factors.

If c is large we may not be able to carry through the process and we still believe that the proposition holds for c. This is due to the fact that at every step the process of division is made up of a definite number of trial multiplications which are only disguised additions, which in their turn reduce to counting. In this way the objective and definite character of the succession of integers is transmitted through addition and multiplication to the constituent problems of division and finally to the problem of division itself. The ultimate source is the definite character of

addition they behave in the same way, viz. $c + (a - a) = c$ for every c and a. Therefore we denote all those differences by the same symbol o. Zero is the first artificial number that enters into our construction of arithmetic.

12. — *Multiplication.* — In our technique of addition we meet sums where all the terms are alike and for convenience we put

(1) $$a + a = 2a,\ a + a + a = 3a, \text{ etc.}$$

Then we learn the simple cases by heart (multiplication table) and develop a technique for products of large numbers based on the definitions (1). In this case we justify the technique by means of our conventions (1) and by the established technique of addition.

Also the rules of manipulation $ab = ba$, $a(bc) = (ab)c$, $a(b + c) = ab + ac$ and, if $b > c$, $a(b - c) = ab - ac$ are « proved » by giving the meaning of products in terms of addition and subtraction. For instance $c = ab$ is interpreted as the sum of a terms all equal to b i. e. in the concrete interpretation : if $c = ab$ then the objects of C can be split into similar groups, everyone of them containing b objects, and the number of the groups is a. To prove $ab = ba$ we have to show that in this case the objects of C can be regrouped into groups containing a objects and the number of these groups will be b. Now if in the first grouping we take an object from each group, we obtain a group of a objects, and we can repeat this operation exactly b times since every group contains precisely b objects. Similar arguments will establish the other two rules of manipulation. In the extension of these rules to our technique of multiplication we have to add that this technique leads to the right number, a statement we had to prove when justifying our technique.

Zero needs special consideration since it is not a « natural » number. From $oc = (a - a)c = ac - ac$ we see that, if we stick to the bracket law, zero obtains a particular property, viz. $oc = o$ for every c. On the other hand, if this property is assigned to o, all integers together with o, will satisfy our rules of manipulation i. e. the introduction of such a zero has no disturbing effect on the operations of arithmetic. Also a fairly convincing direct interpretation of zero can be given by saying that if we add nothing to c, (or subtract nothing from c) c will be unaltered, and if we take

for, in counting, the nature of objects is irrelevant. Thus the systematic nature of the decimal system will extend our initial results on single digits, $a + b = b + a$, say, to sums of any two numbers because at every step the operation will be identical with one of our well established initial operations. In this way we « prove » that the mechanism applied to *a* and *b* and to *b* and *a* lead to the same number.

Also, this number is the right one, since at the first step we count marks corresponding to certain objects of C and C′, then we count their groups of ten and so on until we exhaust the united collection.

This mechanization of counting by means of the decimal notation and by the technique of addition not only splits our work into easy steps but also helps in understanding the principles involved. For example if *a* and *b* are too large for the actual computation of their sums, we still consider $a + b$ as a definite problem with a definite, though unknown, answer because at every step the previous work and the next digits determine the following simple operation (on single digits), i. e. because $a + b$ indicates a finite chain of definite (and very simple) operations. The objective and definite character of simple sums is thus transmitted to sums of large numbers, and as we cannot think of the succession of integers as finally terminated, we say that $a + b$ is a definite *problem* with a definite answer *for any integers a and b.*

11. — *Subtraction.* — If we actually split the objects of a collection C, of number c, into two distinct collections A and B containing *a* and *b* objects respectively, we have $a + b = c$. If we take away the objects contained in A, say, the number of the remaining objects will be *b*. This operation may be indicated by $c - a$ in which case $c - a = b$. The technique of subtraction has been constructed to « calculate » *b* when we know *a* and *c*, without counting the objects in B. This technique is again based on the systematic structure of the decimal notation and on actual subtractions in easy cases when $c < 20$. This operation can only be carried out if $c > a$.

A special feature of subtraction is that it introduces 0 to denote complete exhaustion of collections. The operations $1 - 1, 2 - 2, 3 - 3, \ldots, a - a$ possess the common features (i) they completely exhaust the collection in question, (ii) in our formal technique of

CHAPTER III

ARITHMETIC

10. — *Sums of integers.* — If a and b are the number of objects, all different, contained in the collections C and C′ respectively, and if a + b denotes the number of objects in the united collection (C, C′) obtained by continuing the counting of the objects of C by that of C′ we know that b + a also denotes the same number, for the order of the objects does not affect their number. For the same reason (a + b) + c and a + (b + c) denote the same number. This appeal to the properties of counting establishes the commutative and associative rules of addition for sums of any number of terms.

In mathematical practice, however, the meaning of a + b is slightly different. We count the objects of C and C′ *separately* and from the two numbers so obtained we « calculate » the number of the objects in (C, C′) *without counting them.* This mechanism we substitute for counting the objects of the united collection needs some justification. Also in geometry when we « calculate » the area instead of counting the squares that cover it, this reduction of direct area-measurement to that of lengths followed by some calculation is established step by step for more and more complicated figures.

The well known technique of addition is first established for two, three, fourfold sums of single digits, in which case we can actually verify that it leads to the right number and that these sums are commutative and associative. Then we notice that if in sums of single digits like **2 + 5 + 7 = 14** the numbers indicate tens (hundreds etc.), the result will be **14** tens (hundreds etc.),

relationships between objects, and statements express the re-instatement of attributes into their objects. Thus the statement, this fundamental « logical » function of intellect, appears as the synthesis of the dialectical polarity object-attribute.

We remark finally that mathematicians, realists or formalists, only try to « define » integers in all earnest when they attempt to define « set » in such a way that it should cover transfinite and finite sets.

figure as so many meaningless symbols used as pegs to hang groups of properties on. If also the predicates are considered devoid of meaning, the construction reduces to playing with concrete symbols like letters, $+$, $\vee$, $\supset$ etc. in a concrete way, and the limitations in this game as in playing cards are determined (i) by the concrete physical properties of the symbols used, (ii) by further conventional limitations called the rules of the game. In this extreme case of abstraction, consistency reduces to that of properties of certain material objects used in the game. If we dissociate the meaning from a symbol, the symbol will reassume its concrete existence, i. e. if abstract formalism is pushed to the extreme, it becomes entirely concrete, which is a neat example for the dialectical connection (polarity) between opposites.

It is in this way that Hilbert, the leader of extreme formalists, has practically become a realist. We may say indeed that the realist starts with the concrete « game » of pairing objects of collections and, in particular, marks 1 of his standard collections, and there is no reason why a formalist should not accept this concrete game on its face value.

However, like every concrete activity, this concrete game of counting has got its definite physical limitations, and thus the *endless* succession, even in its mildest form as a devolving succession, cannot be founded on a purely concrete game with meaningless concrete symbols. It requires an insight into the limitless character of our necessarily limited experiences when considered in their succession or co-existence, and it is this intuition which has been crystalized in the idea of integers. Thus integers seem to defy formal definitions, and like any other piece of reality, they can only be « defined » by pointing them out.

We add that in the formalist construction even if the predicates retain their meanings, or some meaning, the position remains very much the same, for by hypothesis there is no object to select and reject the attributes. If the expression « plane figure » is only considered as a peg for attributes, in which case it should really be replaced by x, there is no objection to assigning to it the properties round and square.

The fact is that the separation of attributes from their objects is not final but only « functional », needed for the recognition of

mathematics is thus satisfied if he succeeds in making ideas as clear as that of the integers. This is, in fact, his ultimate aim.

10. — *The formalist attitude.* — If we assign truth to a set of statements, any two of these statements are, by hypothesis, true together and thus they cannot be regarded in the same breath as contradictory, i. e. the problem of their consistency disappears. Formalists, however, are not satisfied with this empty statement and are trying to find a test for consistency in the formal structure of statements.

For this purpose statements are analysed into subject and predicate, and the attribute expressed by the predicate is assigned to the object represented by the subject of the sentence. If the same subject figures in several sentences, the corresponding object will possess several attributes.

Now, the prototypes of object are the material objects in our environment. Any such real object refuses to have two different shapes, weights, colourings, etc. at the same time, but real objects as a whole admit variations within groups like shapes, weights, etc. These groups are fairly indifferent to one another as a definite shape is found associated with various weights and colourings. Also the attribute red entails the attribute coloured. Thus for real objects attributes may be indifferent, exclusive or inclusive, but for a definite real object its own attributes are not mutually exclusive, so that the group of its attributes can be qualified as consistent. This is the ultimate source of our conception of consistency and, as we have seen, the ultimate reason the realist accepts (E) as consistently true is that they describe properties of definite things, viz. those of collections of definite objects.

When we extend this notion of object to objects of thought of our own make, the consistency of the attributes assigned to them becomes a problem, and a rather important problem because, although in a loose sense anything we think of may be said to be an object of thought, without the consistency of their properties they cannot be regarded as proper objects to which grammatical and logical categories can be safely applied.

In the formalist conception the subjects of the various sentences in the given set are considered as exclusively defined by the proposed statements, and therefore, in the construction, they

In this connection it is important to notice that measurement involves counting but counting does not involve measurement. Moreover there are no loopholes in the fact that there are more than two people in a car, if I verify it. However this definite character of facts expressed in the statements of (E) does not « prove » but only makes it possible that no two statements of (E) exclude each other. The ultimate reason for our belief in their consistency must be found in their content.

The contents of the statements of (E) are related to one another in the following way. Table (E) is obtained from $1 < 2$ by (i) successively increasing the larger number by one, which leads to the first row, (ii) increasing both numbers by one, which leads from the first rwo to the other rows. Suppose now that in the pairing process a is exhausted and b is not, i. e. $a < b$. If we add a new object to b, the old pairing will exhaust a again and it will not exhaust the increased collection that we shall denote by $b + 1$. Also, if we add a new object to both a and b, the old pairing completed by pairing the new objects, will exhaust $a + 1$ but not $b + 1$.

The two decisive aspects of this argument seem to be (i) that the statements, i. e. their contents are produced one by one by a definite process viz. by increasing the larger collection or both collections by a single object, (ii) that the process is independent of the size of collections already obtained, i. e. that the process can be repeated *ad lib.* The first part describes the most general features of our experiences in grouping definite objects, and the second is based on our fundamental intuition of the succession of integers. Such a devolving type of argumentation is called a mathematical induction.

For a realist the position is thus roughly this. He accepts an initial square in (E) as undeniably and consistently true if it only refers to collections where the pairing can actually be carried out. By accepting the endless succession of integers as a devolving sequence he also accepts any devolving continuation of the table as undeniably and consistently true. This amounts to saying that if the idea of the devolving endless succession of integers is consistent, so is (E). As a matter of fact, (E) only brings out certain features of this succession. A realist in

of (E) are generally accepted as undeniably true and the corresponding statements in ($\bar{E}$) as undeniably false. The realist, and with him practically everybody, extends this belief to the statements obtained by any extension of the square into a larger square, in which case at every step we only obtain undeniably true statements. Then, of course, no two statements of (E) can be contradictory since, in fact, they are true at the same time, and no two statements of ($\bar{E}$) can be contradictory since, in fact, they are false at the same time so that both (E) and ($\bar{E}$) appear as consistent (separately).

This argument however is too formal, for, in reality, two statements may exclude each other even when they are both « true to fact ». For ex. both statements « the sum of the three angles of a triangle is 180° » and « the sum of the three angles of a triangle is $180 - \frac{1}{10^{1000000}}$ degrees » are equally supported by actual facts although they cannot be true at the same time. The definite limitations in the accuracy of measurements forces us to more or less arbitrary interpolations and extrapolations justified *but not proved* by the successful outcome of expectations based on these hypothetical elements in our theory. Therefore, if the statements of (E) are only « true to fact » in this experimental sense, it may still happen that two of them cannot be true at the same time. Thus, either the form or the content of these statements must tell us that they *are* compatible. For the realist the content makes the statement and thus he will examine the meaning of the statements of (E).

Now, the loopholes in an experimental verification result from the inaccuracy of measurements, and thus facts not based on measurement may be unquestionably true. For instance it would be hard to deny that material objects display some kind of space-time relationship, although the details of this relationship obtained by measurement are only accurate within the interval of precision of our measuring instruments. We cannot deny that statement because it only expresses the fact that there are different objects in our environment, which ultimately amounts to saying that the phenomenal world is not a homogeneous, inarticulate sameness. But, as we have seen, integers i. e. counting is based exactly on this undeniable fact.

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614

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III

LOGIC OF ALGEBRA

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PREFACE

1. — In this tract an attempt is made to clear the ground for a realist discussion of the « crisis » and especially the antinomies in mathematics by an examination of the logical structure of arithmetic and algebra. A systematic description of this structure is, of course, beyond the scope of this short essay. In a second tract the analysis will be extended to the constructions and processes of the Calculus with special consideration to the idea of limit and allied notions including the fundamental conceptions of the theory of sets. The point of view adopted in this work is decidedly realist as represented by Borel or Brouwer and by the new soviet school of philosophy of mathematics, though the influence of Borel's ideas will only come to the fore in the second tract. Our experiences gathered in human practice are the ultimate source of knowledge and there is no higher authority for laying down the laws for science.

Attention is focussed on the part played by the three logical principles and by inference in the construction of algebra, and these principles and the method of inference will emerge in an appropriate shape from mathematical practice, especially from the problems of consistency of algebra, the definiteness of its problems and the technique of proof.

In fact, from the logical point of view, mathematics and in particular algebra is a restricted system of statements with a definite technique for drawing conclusions from given initial statements, and the initial statements together with their conclusions are taken for true. Therefore, if from our premises we can draw

contradictory or mutually exclusive conclusions a and b, both a and b will be considered as true, for they are legitimate consequences, and a will be false because it contradicts b (and *vice versa*). Thus lack of consistency means that in the system the principle of contradiction breaks down. Moreover, if there are questions in the system admitting a third possible answer besides yes or no, the principle of excluded middle does not work in the system.

In such investigations we want to know if a statement is or is not a consequence of the initial statements irrespective of its « truth value », and thus inferences based on truth value like Russell's material implication or Lewes' strict implication are impracticable for our purpose. Classical inference is more suitable for such work because it is based exclusively on structural connections between statements of a certain type, and, as it has been extracted from common place argueing, traditional inference is nearer to mathematical argumentation than more abstract schemes where implication strides over the possible conflict between two true, or two false, statements.

In traditional logic, however, the treatment of inference is not satisfactory because (i) the theory is split into two loosely connected parts viz. immediate inferences and syllogisms, (ii) a satisfactory account of immediate inferences and of the underlying principles is not given. Therefore in Chapters IV and V under the name of logic of collections I give a unified account of the central part of classical logic in the extension interpretation.

I only disagree with Brouwer in two points. In the first place I think we have to go a good deal further than Brouwer in the remodelling of the principles of logic. Connection and derivation of statements are determined by their meaning. Statements on parental relations will display a characteristic network of connections radically different from the connections between statements on collections, and, in Leibniz' terminology, our task is to find an adequate characteristica for every group.

It follows that the meaning of the three principles, if valid at all, as well as the technique of inference will have to emerge from the examination of such groups. A final logic of statements dictating the law to all kinds of statements is a purely theological desideratum. Facts cannot be overruled by abstractions and

concepts are but instruments in the coordination of facts according to their own groupings and lines of division as displayed in our experience.

In the second place, at the suggestion of my belief in the objective character of certain mathematical facts, so strongly emphasised by Brouwer, I want to shift the stress from Brouwer's distinction between known and unknown at present to the difference between definite and indefinite problems, the character of definiteness being transmitted step by step from problems on integers to new problems. In particular, I cannot consider Brouwer's interpretation of his famous example as relevant.

In my work I have found the dialectical method very useful in throwing light on the relationship between opposite notions. This method however is so little known and used that I decided to present the subject in a conventional, non-controversial form. Only some remarks here and there will betray the dialectical origin of some of the ideas.

The formalist outlook is only introduced occasionally as an instructive contrast to realist conceptions, and no systematic criticism is attempted. As a matter of fact a good deal of useful work has been done by formalists, and recent publications by Hilbert's group show a definite appreciation of Brouwer's conception of the principle of excluded middle. (See Gentzen in the bibliography at the end of this tract).

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CHAPTER I

THE NOTION OF INTEGERS

2. — Mathematics is based on the notion of integers. Fractions, negative, real, complex numbers and their generalizations are all defined in terms of integers. Calculus is based on the notion of limit derived from the endless character of the succession of integers. Therefore we begin our analysis of mathematical notions with the analysis of integers. We group our remarks under five headings.

3. — *Direct comparison of collections.* — In our everyday life we constantly come across groups or collections of more or less definite objects like our fingers, the persons in the room, the marks on a tape, the ticks of a clock, etc. Sometimes we pair the objects of two given collections one by one until we exhaust one or both collections and then we declare accordingly that one collection contains more objects than the other or that they contain the same number of objects. This seems to be the fundamental activity that leads to the idea of number. Hilbert also admits the empirical (anschauliche) origin of whole numbers.

In many cases, especially if the objects in question are marks or fairly rigid bodies, we can repeat the pairing in a different order and thus we notice that the result viz. that one of the two collections contains more objects than the other, or that they contain the same number of objects, is independent of the order in which we take the objects. Also if we replace an object in one of our two collections by *any other* object (not taken from the other collection), the outcome of the pairing process will remain the same.

Therefore we can apply the process as soon as the objects of the two collections are definite. Our emotions during the day do not form a collection of objects sufficiently definite for the pairing process. Thus there are definite limitations in the application of pairing.

The other obvious restriction is that we must be able to carry through the pairing. If however for two given collections *a* and *b* these two conditions are realised viz. (a) the objects of *a* and *b* are definite, (b) we can carry through the pairing, the result is one of three possible cases indicated symbolically by $a < b$, $a > b$, $a = b$. Under the above conditions these cases are mutually exclusive and one is always the case.

Thus the notions of more numerous, less numerous, equally numerous involve the idea of « definite object » and the idea of collection of such objects. We also notice that these notions have been derived from a concrete handling of objects for a specific experimental purpose viz. to exhaust the two collections by pairing.

4. — *Indirect comparison of collections. Counting.* — If in the exhaustive pairing of the objects of the collections *a* and *b* and *b* and *c* (leading to $a = b$ and $b = c$) a_i is paired with b_i and b_i with c_i, the pairing (a_i, c_i) establishes an exhaustive pairing of the objects of a and *c* leading to $a = c$. This transitive property of equality extends to inequalities in the sense that if $a < b$ and $b < c$ then $a < c$. In fact $a < b$ means that in the process of pairing the objects of *a* are exhausted and some objects of *b* remain without a pair; similarly for $b < c$. Hence, again (a_i, b_i) and (b_i, c_i) lead to (a_i, c_i) with some of the objects of c remaining unpaired.

This transitive property coupled with the fact that no particular feature of the objects can effect the outcome of the pairing leads to the establishment of a kind of gold standard for the comparison of collections. In fact, instead of directly comparing two given collections we compare both with collections of standardised marks like **1** made on a tree or on a sheet of paper. A scale of standard collections is constructed (a) by continued grouping of marks **1** into larger and larger collections, (b) by systematic naming of these standard collections **1**, **11**, **111**, **1111**, ... leading to the decimal notation, say.

The pairing of the objects of a collection a with standard marks, 1, results in an exhaustive pairing of a with one of our standard collections and then the name of this standard collection is affixed to a as a label and is called the *number* of its objects, and the process is then referred to as *counting* the objects of a.

This number of the objects of a collection designates a property of the collection in relation to other collections. Counting is *our way* of obtaining this piece of information about a given collection but counting is only an enquiry the outcome of which depends on the collection itself.

This objective factual character of numbers induces us to believe that as soon as the objects of a collection are all definite, so is their number irrespective of our counting them. If, in reality, we can carry through the counting but we are just too lazy or indifferent to do it, this conception is justified enough. If, however, we are unable to carry through the process even with the help of some machines, the meaning of our belief becomes dubious. To make the meaning clear we have to examine the case when condition (3) (b) is dropped. This problem is closely connected with the idea of endless succession of standard collections.

5. — *The idea of endless succession.* — The statements of arithmetic, especially the endlessness of the succession of integers, are as a rule considered as undeniable and in arithmetic no radical change has ever been contemplated. In every other science we not only admit a certain element of doubt but we take development for granted. Is then arithmetic essentially different from other sciences?

Scientific theories are based on the following process. If we know the probable outcome of our actions, i. e. if we know the habits of objects around us, we have some chance in choosing the action that will change our environment in the way we desire it. When the number of tabulated facts (habits) of objects becomes too extensive for our memory we complete them into strings (graphs) and unite them into more or less abstract models (theories) handled by our intellect rather than by our memory. Thus a scientific theory is an abstract instrument constructed for coordinating facts and for putting new questions to nature. The

answer to these and other questions will need a fresh coordination and the cycle starts again.

The only essential condition for such a construction is success. If, however, the theory leads indifferently to two contradictory answers we cannot very well use it for forecasting events. Thus consistency *is a practical requirement* in the construction of scientific models.

Now, arithmetic also is founded on facts and it seems to go beyond the facts (i) when a number is attached to collections inaccessible or too large for an actual pairing process, (ii) when the succession of integers is thought of as endless. Thus arithmetic appears at first as an abstract model of the usual type constructed for the coordination of our experiences, on pairing the objects of two collections, by means of the usual extrapolation; i. e. by an abstract continuation of concrete facts.

Step (i) only concerns the application of arithmetic and involves the usual risk inherent in the application of any instrument.

Step (ii) is suggested, although not absolutely required by the progressive nature of our knowledge whose development entails the introduction of larger and larger collections of observed or assumed bodies like stars, atoms, electrons. This suggestion amounts to keeping the succession of numbers *unfinished*.

Why do we believe then that we not only want this succession to be kept unfinished, but that we cannot help thinking of it as unfinished?

Now, also in geometry and in physics we admit without the shadow of a doubt that bodies appear in various configurations and successions, i. e., that they are related somehow in space and time, and our doubt only concerns our specified statements about the positional or space-time relations of definite classes of bodies as expressed for ex. in Euclid's axioms or in Newton's laws of motion. Since, with our body and mind, we are part and parcel of the world, we cannot help experiencing certain features of this world, in particular the fact that our phenomenal world consists of various things, that this world is not a featureless homogeneous one-ness. The articulations and alignments of these features may be difficult to perceive and their description may need arbitrary, hypothetical elements subject to doubt, but there can be no doubt as to the existence of features.

Now the idea of the succession of integers seems to hinge solely *on the mere existence of different things*, it seems to describe the most general (and the most superficial) aspect of our world. In particular, step (ii) i. e. abstract continuation of our creation of larger and larger collections, is based on the fact that our mind, a part of the phenomenal world, also displays its general structure viz. it is not a homogeneous one-ness. If we think of some objects (of thought) as a collection, the very thought of their collection will be different from that of the individual objects, and this idea of their collection as a definite new object (of thought) can be joined to the collection increasing its number by one. This mental experiment « proves » that we cannot help thinking of this process as essentially unfinished, open to further continuation. This insight into the character of the process of forming larger and larger collections may be described as the *intuition of an endless succession* where the successive steps are telescoped into one another to form the aggregates of steps already accomplished.

From the dialectical point of view we may say that the very fact that every human act is limited, and thus every event we are able to know is also limited, implies the unlimited nature of the succession of such limited acts or events. In fact, if every actual collection or succession is limited, there is something else that limits it, and thus the succession can have no final termination. « Limited » is really unthinkable without its dialectical opposition « unlimited », and the endless succession of integers is the unfolding of this dialectical polarity.

6. — *The objectivity of integers.* — When we actually count the objects of a group, we try to pair these objects with the marks of our successive standard collections, and the outcome of this trial depends on the group and not on us. In this sense the number is a property of the group. If we substitute a new object for any of the given objects, there is an obvious exhaustive pairing of the old and new group viz. pairing indentical elements together and pairing the substituted object with the object it replaces. This shows that the number of the objects in a group is independent of the actual nature of the objects provided that they are definite.

Both remarks apply to abstract objects. Moreover concrete

and abstract objects, if they are definite, can replace one another without altering the number so that for the process of counting there is no difference between real and mental objects. This is due ultimately to the fact that our mind is a part of the phenomenal world and the process of counting is based on general features of this world displayed in every part of it.

Therefore we accept Brouwer's sharp distinction between *mathematical facts* we may or may not observe and *mathematical language* in which we couch observed mathematical facts or questions. In particular, we accept that there is a fundamental fact, an underlying intuition that justifies the construction of the endless succession of integers as definite objects (of thought) with definite properties.

7. — *The class of integers.* — If we want to keep the succession of integers unfinished, or if we cannot help thinking of it as unfinished, the succession of individual integers will necessarily be thought of as *inexhaustible* by human acts or thoughts. This conception rules out the idea of *their totality as unthinkable*, a conclusion which assumes an exceptional importance when dealing with antinomies about the « all ».

On the other hand, the class of integers seems to be a definite notion whose limits are rather less vague than those of most practical or theoretical classes. This is due to the fact that the individual integers are produced one after the other by a *definite* process and the definite nature of this rule of construction delimits the class in a sufficiently clear manner. The individuals of the class are not all there, they are only collectively characterised by the machine that produces them. We shall say that they form an endless *devolving sequence*, devolving according to the given rule.

Therefore, in a sense, integers form a *closed* community, the closure being produced by their way of generation. We may even say that their class is *complete* in so far as nothing is admitted into the class unless generated by the given process, so that anything we want to call an integer is contained in the class and also every individual generated by the process is admitted to the class, so that the class only contains integers.

As a matter of fact a good many of our usual « classes » are of the same nature. The fact that millions of insects are continually

born (and others die), new pieces of furniture are continually constructed (and old pieces destroyed) does not seem to affect the definiteness of our idea of the class of insects or of furniture, because here again we think of the class as determined not so much by its existing objects as by the structure or construction of its objects.

A class is thus more often determined by a general description of the objects in view e. g. by attributes like red, round, weighing one pound, people living in London, than by the enumeration of the objects. In such cases the idea is made clear by some samples and by adding « objects like these » or by the description of their structure or way of generation (family). The totality of objects possessing the attributes in view is irrelevent for the understanding of the idea which can be made clear and definite without any reference to « all » objects of the same attribute. We shall reserve the name *class* for such « like this » or attribute definitions (or rather delimitations) of objects, and will use *collection* where the objects alone determine the grouping e. g. a sack of peas, a display of miscellaneous objects for sale, the books in a library etc. When a collection has been formed, we can always construct an attribute e. g., the books in the catalogue, defining the collection formally as a class, but in general, an attribute definition cannot be transformed into a collection definition. The main difference between the two attitudes is that in a collection the objects come first, their assemblage afterwards, whereas in a class the link, the common attribute, comes first, the objects afterwards (they may even be missing).

CHAPTRE II

INEQUALITIES BETWEEN INTEGERS AND THEIR CONSISTENCY

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8. — *The problem.* — We begin our analysis of the idea of integers sketched in the previous section by the discussion of the consistency of statements involved in the construction. As we have already noticed, consistency is a practical requirement of scientific theories, and as a rule it is « established » by a reduction to the consistency of mathematics. Hence the central importance of our problem. We notice however that even inconsistent theories may be of some advantage.

A set of statements is usually considered as *consistent* if it satisfies the two conditions : (i) the set does not contain contradictory statements, (ii) the conclusions drawn from the set do not contradict one another or the statements of the set. In the current usage two statements are said to be *contradictory* if they can be neither true nor false at the same time. They are *contraries* if they cannot be true but may be false at the same time.

In our discussion of the problem of consistency we shall make use of the ordinary notation. If m is the number of marks in a standard collection, the addition of another mark to the collection is indicated by $+ 1$ (one more mark) and the corresponding number is denoted by $\mathrm{m} + 1$. The fact that the number of the increased collection is larger than that of the original collection is denoted by $\mathrm{m} < \mathrm{m} + 1$ or by $\mathrm{m} + 1 > \mathrm{m}$ (*inqualities*). Thus we obtain the devolving endless sequence of inequalities

(e) $$1 < 2,\ 2 < 3,\ 3 < 4,\ \dots$$

For a clearer understanding of the idea of false statements involved in the notion of contradiction consider the four statements « the earth is round », « 3 is a lucky number », « $3 < 2$ », « $2 < 4$ ». In one respect they are in the same position relative to (e), viz. none of them figures in it. Their respective position in relation to (e) differs inasmuch as the first states nothing about integers, the second states an emotional attitude toward an integer, the third uses the symbol $<$ in the wrong way, the fourth is a *consequence* of $2 < 3$ and $3 < 4$ if we admit the transitive property of $<$.

The set (e) is *indifferent* to the first two statements, for they say nothing about the « greater or less » relation between two numbers; it stamps $3 < 2$ as *false*, for, in the accepted interpretation of the symbols 3, $<$ and 2, a collection that contains 3 objects is more and not less numerous than a collection that contains 2. We need the statements

$$(\bar{e}) \qquad 1 > 2,\ 2 > 3,\ 3 > 4,\ \ldots$$

in problems where we have to consider the various possible cases. In fact, if we use the letters m and n to denote the number of two collections, the process of pairing may result in any of the three statements $m < n$, $m > n$, $m = n$. We notice, however, that $(\bar{e})$ is not incorporated in the structure of arithmetic. It is only used as a kind of scaffoldage in the construction.

The fourth statement, $2 < 4$, is admitted as a part of arithmetic. This idea of consequence involves a definite mechanism of inference, based here on the transitive property of $<$, and leads to the question whether mathematical inference is identical with inference by syllogism. In order to avoid for the moment the difficulties inherent in this problem, we complete (e) and $(\bar{e})$ into the devolving tables

$$(E)\ \begin{matrix} 1 < 2, & 1 < 3, & 1 < 4, & \ldots \\ 2 < 3, & 2 < 4, & 2 < 5, & \ldots \\ 3 < 4, & 3 < 5, & 3 < 6, & \ldots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \end{matrix} \qquad (\bar{E})\ \begin{matrix} 1 > 2, & 1 > 3, & 1 > 4, & \ldots \\ 2 > 3, & 2 > 4, & 2 > 5, & \ldots \\ 3 > 4, & 3 > 5, & 3 > 6, & \ldots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \end{matrix}$$

without raising the question if (E) and $(\bar{E})$ give us *all* the consequences of (e) and $(\bar{e})$ respectively.

9. — *The realist attitude.* — The statements in an initial square

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